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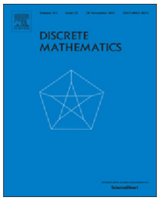

Note

Characterizations of graphs G having all $[1, k]$-factors in kG

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ABSTRACT

Let $k \geq 1$ be an integer and G be a graph. Let kG denote the graph obtained from G by replacing each edge of G with k parallel edges. We say that G has *all* $[1, k]$*-factors* or *all fractional* $[1, k]$*-factors* if G has an h-factor or a fractional h-factor for every function $h : V(G) \to \{1, 2, \ldots, k\}$ with $h(V(G))$ even. In this note, we come up with simple characterizations of a graph G such that kG has all $[1, k]$-factors or all fractional $[1, k]$-factors. These characterizations are extensions of Tutte's 1-Factor Theorem and Tutte's Fractional 1-Factor Theorem.

1. Introduction

All graphs considered in this paper are *multigraphs*, which may have multiple edges but have no loops. A graph having neither loops nor multiple edges is called a *simple graph*. For convenience, we simply call a multigraph a *graph* when we give definitions and notations. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let kG denote the graph obtained from G by replacing each edge of G with k parallel edges. The number of vertices of G is referred to as the *order* of G and denoted by $|G|$. We denote the degree of a vertex v in G by $d_G(v)$. For two disjoint subsets $S, T \subseteq V(G)$, let $e_G(S, T)$ denote the number of edges of G joining S to T. For a set X, we denote the cardinality of X by $|X|$. A vertex of degree zero is called an *isolated vertex*. Let $Iso(G)$ denote the set of isolated vertices of G, and let $iso(G) = |Iso(G)|$. Let $\omega_{\geq k}(G)$ denote the number of components of G with order at least k and let $\omega(G) = \omega_{\geq 1}(G)$. For a vertex x of G, $N_G(x)$ denotes the set of the vertices adjacent to x in G. For a subset $X \subseteq V(G)$, we write $N_G(X) = \bigcup_{x \in X} N_G(x)$.

Let $\mathbb{Z}^+$ denote the set of non-negative integers. Let $g, f : V(G) \to \mathbb{Z}^+$ be integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. A (g, f)*-factor* of G is a spanning subgraph F of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$. If $g(x) = f(x)$ for all vertices $x \in V(G)$, then a (g, f)-factor is called an f*-factor*. Let $k \geq 1$ be a fixed integer, then a $[1, k]$*-factor* is a (g, f)-factor with $g(x) \equiv 1$ and $f(x) \equiv k$ for every vertex x. For a real-valued function $w : E(G) \to [0, 1]$, we write $E_{w>0} = \{e \in E(G) \mid w(e) > 0\}$. If an edge e is incident with a vertex x, then we write $x \sim e$ or $e \sim x$. For given functions g and f, if $g(x) \leq \sum_{e \sim x} w(e) \leq f(x)$ holds for every $x \in V(G)$, then the spanning subgraph $F = (V(G), E_{w>0})$

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is called a *fractional* (g,f)*-factor* of G with indicator function w. If no confusion can arise, we briefly call w a *fractional* (g,f)*-factor* of G. When $g(x)=f(x)$ for all $x \in V(G)$, a fractional (g,f)-factor is referred to as a *fractional* f*-factor*. Clearly, a (g,f)-factor is a fractional (g,f)-factor w satisfying $w(e) \in \{0,1\}$ for every $e \in E(G)$ and vice versa.

For a function f defined on $V(G)$ and a subset $X \subseteq V(G)$, we write $f(X) := \sum_{x\in X} f(x)$. For two functions $g, f : V(G) \to \mathbb{Z}^+$ with $g \le f$, define

$$\mathcal{H}_{g,f} = \{h : V(G) \to \mathbb{Z}^+ \mid g(x) \le h(x) \le f(x) \text{ for all } x \in V(G)\}$$

and

$$\mathcal{H}^{even}_{g,f} = \{h : V(G) \to \mathbb{Z}^+ \mid g(x) \le h(x) \le f(x) \text{ for all } x \in V(G), \\ \text{and } h(V(G)) \text{ is even}\}.$$

Then we say that G has *all* (g,f)*-factors* if G contains an h-factor for every $h \in \mathcal{H}^{even}_{g,f}$. If, for every $h \in \mathcal{H}_{g,f}$, G contains a fractional h-factor, then we say that G has *all fractional* (g,f)*-factors*.

Tutte (1947) gave sufficient and necessary conditions for a simple graph to have 1-factors.

Theorem 1.1 (*Tutte, [3]*). *A simple graph G has a 1-factor if and only if*

$$odd(G-S) \le |S| \quad \text{for all } S \subset V(G), \tag{1}$$

where $odd(G-S)$ denotes the number of components of $G-S$ with odd order.

For fractional 1-factors, Tutte (1953) obtained the following criterion.

Theorem 1.2 (*Tutte, [4]*). *Let G be a simple graph. Then G has a fractional 1-factor if and only if*

$$iso(G-S) \le |S| \quad \text{for all } S \subseteq V(G). \tag{2}$$

In this note, we characterize a graph G such that kG has all $[1,k]$-factors or all factional $[1,k]$-factors, respectively, and these characterizations generalize the above Tutte's Theorems. The following two theorems are the main results.

Theorem 1.3. *Let $k \ge 2$ be an integer and G be a connected multigraph. Then kG has all $[1,k]$-factors if and only if for every $S \subset V(G)$, we have*

$$k \cdot iso(G-S) + \omega_{\ge k+1}(G-S) \le |S| + 1. \tag{3}$$

Theorem 1.4. *Let $k \ge 1$ be an integer and G be a multigraph. Then kG has all fractional $[1,k]$-factors if and only if*

$$k \cdot iso(G-S) \le |S| \quad \text{for all } S \subset V(G). \tag{4}$$

In the proofs of main theorems, we need the following theorems.

Theorem 1.5 (*Niessen, [2]*). *Let G be a connected multigraph and $g, f : V(G) \to \mathbb{Z}^+$ such that $0 \le g(v) < f(v)$ for all $v \in V(G)$. Then G has all (g,f)-factors if and only if for all $S, T \subseteq V(G)$ with $T \cap S = \emptyset$,*

$$g(S) - f(T) + \sum_{x\in T} d_{G-S}(x) - \omega(G-S-T) \ge -1,$$

where $\omega(G-S-T)$ denotes the number of components of $G-(S\cup T)$.

Theorem 1.6 (*Lu, [1]*). *Let G be a multigraph and $g, f : V(G) \to \mathbb{Z}^+$ such that $0 \le g(v) \le f(v)$ for all $v \in V(G)$. Then G has all fractional (g,f)-factors if and only if for all $S, T \subseteq V(G)$ with $T \cap S = \emptyset$,*

$$g(S) - f(T) + \sum_{x\in T} d_{G-S}(x) \ge 0.$$

Note that many results on factional factors can be found in [5].

2. Proofs of Theorems 1.3 and 1.4

In this section, we prove the main results.

Proof of Theorem 1.3. Necessity (⇒). Suppose that kG contains all $[1,k]$-factors. Since G is connected, the result holds for $S=\emptyset$. So we may assume that $S \ne \emptyset$. Let $q = \omega_{\ge k+1}(G-S)$ and let $D_1, \ldots, D_q$ be the components of $G-S$ with order at least $k+1$. We choose a vertex $v_i \in V(D_i)$ for $1 \le i \le q$ and $u \in S$. Define $h', h'' : V(G) \to \mathbb{Z}^+$ as

$$h'(v) = \begin{cases} k, & \text{if } v \in Iso(G-S); \\ 2, & \text{if } v = v_i \text{ and } |D_i| \equiv 0 \pmod 2; \\ 1, & \text{otherwise,} \end{cases}$$

and

$$h''(v) = \begin{cases} k, & \text{if } v \in Iso(G-S); \\ 2, & \text{if } v = v_i \text{ and } |D_i| \equiv 0 \pmod 2; \\ 2, & \text{if } v = u; \\ 1, & \text{otherwise.} \end{cases}$$

One can see that $h'(V(G)) + h''(V(G))$ is odd. Let $h : V(G) \to \mathbb{Z}^+$ be defined as

$$h = \begin{cases} h', & \text{if } h'(V(G)) \equiv 0 \pmod 2; \\ h'', & \text{otherwise.} \end{cases}$$

So we have $h \in \mathcal{H}_{1,k}^{even}$. By the hypothesis, kG contains an h-factor F. If $h = h'$, then we have

$$\begin{aligned} |S| = \sum_{v \in S} d_F(v) &\geq e_F(S, Iso(G-S)) + e_F(S, \cup_{i=1}^{q} V(D_i)) \\ &\geq k \cdot iso(G-S) + \omega_{\geq k+1}(G-S). \end{aligned}$$

Next assume that $h = h''$. Since kG has an h''-factor F, we have

$$\begin{aligned} |S| + 1 = \sum_{v \in S} d_F(v) &\geq e_F(S, Iso(G-S)) + e_F(S, \cup_{i=1}^{q} V(D_i)) \\ &\geq k \cdot iso(G-S) + \omega_{\geq k+1}(G-S). \end{aligned} \tag{5}$$

Hence,

$$k \cdot iso(G-S) + \omega_{\geq k+1}(G-S) \leq |S| + 1.$$

Sufficiency ($\Leftarrow$). Suppose to the contrary that kG does not have all $[1,k]$-factors. By Theorem 1.5, there exist two disjoint subsets S, T such that

$$\delta_{kG}(S,T) = |S| - k|T| + \sum_{x \in T} d_{kG-S}(x) - \omega(kG - S - T) \leq -2.$$

We choose S, T so that T is minimal. Set $U = V(G) - S - T$. One can see that $\omega(kG - S - T) = \omega(G - S - T)$.

Claim 1. *$T = \emptyset$ or $G[T]$ consists of isolated vertices.*

Suppose that there exists an edge uv in $G[T]$. Let $T' := T - u$. If $e_G(u, U) \geq 1$, then we have

$$\begin{aligned} \delta_{kG}(S,T') &= |S| - k|T'| + \sum_{x \in T'} d_{kG-S}(x) - \omega(G - S - T') \\ &\leq |S| - k(|T|-1) + \sum_{x \in T} d_{kG-S}(x) - k(1 + e_G(u,U)) \\ &\quad - \big(\omega(G-S-T) - e_G(u,U) + 1\big) \\ &= \delta_{kG}(S,T) - (k-1)e_G(u,U) - 1 \\ &\leq \delta_{kG}(S,T) \leq -2. \end{aligned}$$

This contradicts the choice of S, T. If $e_G(u, U) = 0$, then

$$\begin{aligned} \delta_{kG}(S,T') &= |S| - k|T'| + \sum_{x \in T'} d_{kG-S}(x) - \omega(G - S - T') \\ &\leq |S| - k(|T|-1) + \sum_{x \in T} d_{kG-S}(x) - k - (\omega(G-S-T) + 1) \\ &= \delta_{kG}(S,T) \leq -2. \end{aligned}$$

This contradicts the choice of T. Therefore Claim 1 holds.

Claim 2. $E_G(T, U) = \emptyset$.

Assume that there exist $u \in U$ and $v \in T$ such that $uv \in E(G)$. Then we have

$$\begin{aligned}
\delta_{kG}(S, T - v) &= |S| - k(|T| - 1) + \sum_{x \in T - v} d_{kG-S}(x) - \omega(G - S - (T - v)) \\
&\le |S| - k|T| + k + \sum_{x \in T} d_{kG-S}(x) - k \cdot e_G(v, U) \\
&\qquad - \big(\omega(G - S - T) - e_G(v, U) + 1\big) \\
&= \delta_{kG}(S, T) - (k - 1)(e_G(v, U) - 1) \\
&\le \delta_{kG}(S, T) \le -2.
\end{aligned}$$

This contradicts the minimality of T again. Hence Claim 2 holds.

By Claims 1 and 2, we have $\sum_{x \in T} d_{kG-S}(x) = 0$ and $iso(G - S) = iso(G - S - T) + |T|$. Thus

$$\begin{aligned}
-2 \ge \delta_{kG}(S, T) &= |S| - k|T| + \sum_{x \in T} d_{kG-S}(x) - \omega(G - S - T) \\
&= |S| - k|T| - \omega(G - S - T) \\
&= |S| - k|T| - iso(G - S - T) - \omega_{\ge 2}(G - S - T) \\
&= |S| - k \cdot iso(G - S) + (k - 1) \cdot iso(G - S - T) - \omega_{\ge 2}(G - S) \\
&\ge |S| - k \cdot iso(G - S) - \omega_{\ge 2}(G - S).
\end{aligned}$$

It follows that

$$|S| + 2 \le k \cdot iso(G - S) + \omega_{\ge 2}(G - S). \tag{6}$$

We choose a maximal S such that the inequality (6) holds.

Claim 3. *Every non-trivial component of $G - S$ contains at least $k + 1$ vertices.*

Suppose that there is a component D in $G - S$ with $2 \le |V(D)| \le k$. Let $u \in V(D)$ and $S' = S \cup (V(D) - u)$. Then, we have

$$\begin{aligned}
&|S'| - k \cdot iso(G - S') - \omega_{\ge 2}(G - S') \\
=&|S| + |V(D)| - 1 - k \cdot (iso(G - S) + 1) - (\omega_{\ge 2}(G - S) - 1) \\
\le&|S| - k \cdot iso(G - S) - \omega_{\ge 2}(G - S) \le -2.
\end{aligned}$$

This contradicts the choice of S and thus completes the proof of Claim 3.

By Claim 3 and (6), we have

$$|S| + 2 \le k \cdot iso(G - S) + \omega_{\ge k+1}(G - S). \tag{7}$$

This inequality contradicts the assumption (3). Consequently, the proof is completed. □

Proof of Theorem 1.4. By Theorem 1.2, we may assume that $k \ge 2$.

Necessity ($\Rightarrow$). Let $S \subset V(G)$. Suppose that kG contains all fractional $[1, k]$-factors. Define $h : V(G) \to \mathbb{Z}^+$ by

$$h(v) = \begin{cases} k, & \text{if } v \in Iso(G - S); \\ 1, & \text{otherwise.} \end{cases}$$

Then kG contains a fractional h-factor F with an indicator function w. Thus we have

$$\begin{aligned}
|S| = \sum_{v \in S} d_F(v) &\ge \sum_{e \in E_{kG}(S, Iso(G - S))} w(e) \\
&= \sum_{x \in Iso(G - S)} d_F(x) \\
&= k \cdot iso(G - S).
\end{aligned}$$

Sufficiency ($\Leftarrow$). Assume that kG does not have all fractional $[1, k]$-factors. By Theorem 1.6, there exist two disjoint subsets S, T such that

$$\delta_{kG}(S, T) = |S| - k|T| + \sum_{v \in T} d_{kG-S}(x) < 0. \tag{8}$$

We choose S and T so that T is minimal.

Claim 1. $E_G(T, V(G) - S) = \emptyset$.

Suppose there exists an edge $uv \in E_G(T, V(G) - S)$ with $u \in T$. Let $T' := T - u$. Then we have

$$\begin{aligned}\delta_{kG}(S, T') &= |S| - k|T'| + \sum_{x \in T'} d_{kG-S}(x) \\ &\le |S| - k(|T| - 1) + \sum_{x \in T} d_{kG-S}(x) - k \cdot e_G(u, V(G) - S) \\ &\le \delta_{kG}(S, T) < 0,\end{aligned}$$

contradicting the choice of T. This completes the proof of Claim 1.

By Claim 1, $T \subset Iso(G - S)$. The inequality (8) and Claim 1 imply

$$\begin{aligned}0 > \delta_{kG}(S, T) &= |S| - k|T| + \sum_{v \in T} d_{kG-S}(x) && (9) \\ &\ge |S| - k \cdot iso(G - S), && (10)\end{aligned}$$

namely,

$$k \cdot iso(G - S) > |S|.$$

This contradicts (4). Therefore the proof is completed. □

Remark: For graphs having all (g, f)-factors or all fractional (g, f)-factors, their characterizations are known (i.e., Theorems 1.5 and 1.6). However, in this note, we provide much simpler criteria for graphs to have all $[1, k]$-factors or all fractional $[1, k]$-factors in terms of isolated vertices. The criteria only use single subset S of $V(G)$ much like that in Tutte's 1-Factor Theorem, rather than examining all pairs of disjoint vertex sets. The simple criteria will be helpful to yield structures of graphs containing such factors and thus obtain algorithms to identifying these factors.

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Declaration of competing interest

We declare that we have no conflicts of interest to this work. We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

References

[1] H.L. Lu, Simplified existence theorems on all fractional $[a, b]$-factors, Discrete Appl. Math. 161 (2013) 2075–2078.
[2] T. Niessen, A characterization of graphs having all (g, f)-factors, J. Combin. Theory Ser. B 72 (1998) 152–156.
[3] W.T. Tutte, The factorization of linear graphs, J. Lond. Math. Soc. 22 (1947) 107–111.
[4] W.T. Tutte, The 1-factors of oriented graphs, Proc. Amer. Math. Soc. 4 (1953) 922–931.
[5] Q.L. Yu, G.Z. Liu, Graph Factors and Matching Extensions, Springer, 2009.