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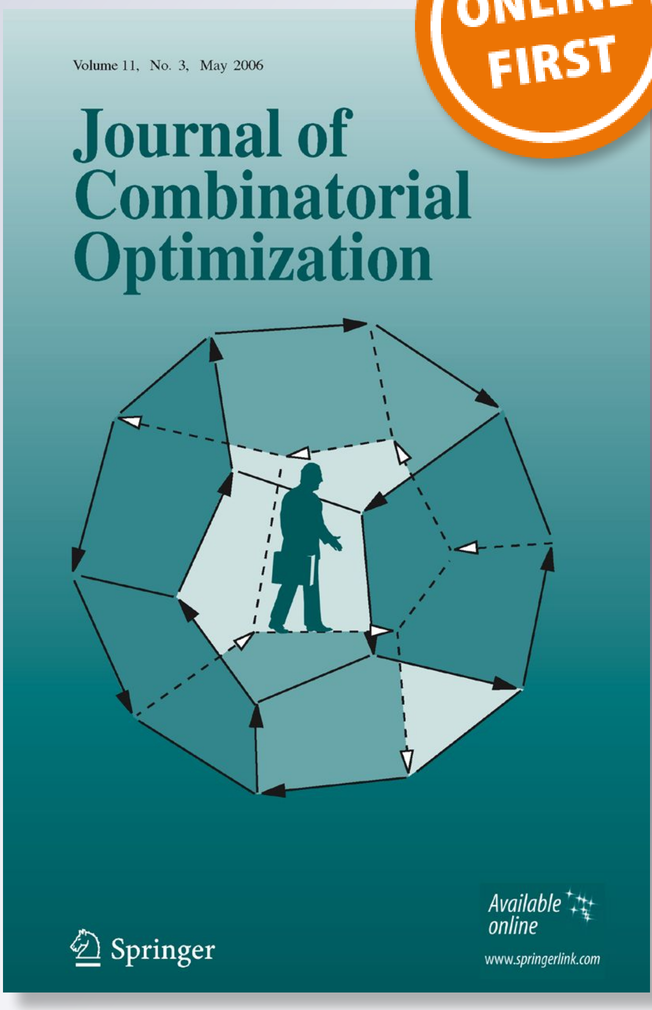

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A polynomial algorithm determining cyclic vertex connectivity of 4-regular graphs

Jun Liang[1,2] · Dingjun Lou[2] · Zongrong Qin[2] · Qinglin Yu[3]

Abstract
For a connected graph G, a set S of vertices is a cyclic vertex cutset if $G - S$ is not connected and at least two components of $G - S$ contain a cycle respectively. The cyclic vertex connectivity $c\kappa(G)$ is the cardinality of a minimum cyclic vertex cutset. In this paper, for a 4-regular graph G with v vertices, we give a polynomial time algorithm to determine $c\kappa(G)$ of complexity $O(v^{15/2})$.

Keywords Cyclic vertex connectivity · 4-Regular graph · Maximum flow · Time complexity

1 Introduction and terminology

All graphs considered in this paper are simple, undirected, finite and connected. We use the notation and terminology of Bondy and Murty (1976). In particular, for a graph G, $v(G)$ denotes the number of vertices of G and $g(G)$ denotes the girth of G, i.e., the length of a shortest cycle of G. If there is no ambiguity, then we write v and g instead of $v(G)$ and $g(G)$.

For a graph G, a set S of vertices (edges) in G is a *cyclic vertex (edge) cutset* if $G-S$ is not connected and at least two components of $G-S$ contain a cycle respectively. The *cyclic vertex connectivity* $c\kappa(G)$ is the cardinality of a minimum cyclic vertex cutset in G. We say that $c\kappa(G)$ is ∞ if no cyclic vertex cutset exists. Note that cyclic vertex cutset is different from cycle-separating vertex cut introduced by McCuaig (1992) and Nedela and Škoviera (1995).

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Let C be a cycle of G. An edge whose ends are both on C is called a *chord* of C if the edge does not belong to the set of edges of C. A cycle is called *an induced cycle* if the cycle does not contain any chord.

Let C be an induced cycle embedded on a plane. Suppose $c = |V(C)|$, $C = a_0a_1 \cdots a_{c-1}a_0$ and $0 \leq i < j \leq c-1$, then we use $C^+[a_i, a_j]$ ($C^-[a_i, a_j]$) to denote the set of vertices from a_i to a_j on C in the clockwise (counterclockwise) direction of C. Furthermore, the symbols $'('$ and $')'$ are used instead of $'['$ and $']'$ if a_i and a_j are not contained in the set $C^+[a_i, a_j]$ or $C^-[a_i, a_j]$. We also use $d_C(a_i, a_j)$ to denote the distance between vertices a_i and a_j on C. Obviously, we have that $d_C(a_i, a_j) = d_C(a_j, a_i)$. Besides, the symbol $a_i \overrightarrow{C} a_j$ (or, $a_i \overleftarrow{C} a_j$) denotes the path $a_ia_{i+1} \cdots a_j$ (or, $a_ia_{i-1} \cdots a_j$) in the clockwise (or, counterclockwise) direction of cycle C. For example, let $C = a_0a_1a_2a_3a_4a_0$, then $a_0 \overrightarrow{C} a_3$ (or, $a_0 \overleftarrow{C} a_3$) denotes the path $a_0a_1a_2a_3$ (or, $a_0a_4a_3$) on C.

Some important work on cyclic edge connectivity were done in Aldred et al. (1991), Kutnar and Marušič (2008), Lou and Holton (1993) and McCuaig (1992), Nedela and Škoviera (1995), Peroche (1983) and Tait (1880). Dvořák et al. (2004) showed that the cyclic connectivity can replace the usual connectivity in applications where the considered graphs have a bounded maximum degree, such as robustness of local computer networks, parallel computer architectures and others. They presented an $O(v^2 \log^2 v)$-algorithm for cyclic edge connectivity of cubic graphs. Then Lou and Liang (2014) and Lou and Wang (2005) gave algorithms determining the cyclic edge connectivity of k-regular graphs, and the time complexity in Lou and Liang (2014) is $O(k^9v^6)$. There was little previous work on algorithm determining the cyclic vertex connectivity. The results obtained by us were an $O(v^{15/2})$–algorithm for cyclic vertex connectivity of cubic graphs in Liang et al. (2017) and an $O(v^{15/2}k^7k^{9k^2})$—algorithm for cyclic vertex connectivity of k-regular graphs in Liang and Lou (2018). In this paper, we find a polynomial algorithm to determine the cyclic vertex connectivity of 4-regular graphs which is a key step for solving the cyclic vertex connectivity problem of k-regular graphs.

The paper is divided into four sections. The first section contains basic definitions, backgrounds, and known results. The second section contains an algorithm (Algorithm 1) which determines the cyclic vertex connectivity of 4-regular graphs and its time complexity analysis. The third section gives some conclusions used to prove the correctness of Algorithm 1. The fourth section proves the correctness of Algorithm 1.

To conclude this section, we list several results which will be used in the proof of the main result in the later section.

Lemma 1.1 (Liang et al. 2017, Theorem 2.3) *Let G be a connected k-regular graph with girth g and v vertices. If $v \geq 2g(k-1)$, then $c\kappa(G) \leq (k-2)g$.*

Lemma 1.2 (Liang et al. 2017, Lemma 3.2) *For any k-regular graph G with girth g, if $v(G) < 2g(k-1)$, then* (1) *if $k = 3$, then $g \leq 10$;* (2) *if $k = 4$, then $g \leq 7$;* (3) *if $k = 5$ or 6, then $g \leq 6$;* (4) *if $7 \leq k \leq 25$, then $g \leq 5$;* (5) *if $k \geq 26$, then $g \leq 4$.*

Lemma 1.3 (Liang et al. 2017, Lemma 3.3) *Let G be a k-regular graph with girth $g \geq 7$, suppose that $C = a_0a_1 \ldots a_{c-1}a_0$ is an induced cycle in a connected component of G, then $|\bigcup_{i=0}^{c-1} N_1(a_i)| \geq g(k-2)$. ($N_r(a_i)$ see Definition 3.2)*

2 An algorithm for finding the cyclic vertex connectivity of 4-regular graphs

In this section, we describe an algorithm for the cyclic vertex connectivity of 4-regular graphs. The idea of the algorithm is that, we find all induced cycles of length at most $4\log_3 2v + x_0$ (x_0 is a positive constant) in G, and apply the maximum flow-minimum cut algorithm to get the minimum cutset between each pair of them, then a minimum cyclic vertex cutset is the minimum cutset. In Algorithm 1, the symbol s denotes the initial value of cyclic vertex connectivity $c\kappa(G)$, and z is a temporary variable.

Algorithm 1

1. For each vertex u in G, use a breadth first search strategy to find a shortest cycle containing u, thus we find the girth g of G; // $O(v^2)$
2. If $v(G) \geq 6g$, then $s := 2g$, else $s := \infty$; // $O(1)$
3. If $g \geq 19$, then $z = 4\log_3(2v) + 7$,

   else $z = 4\log_3(2v) + 42$;

4. For each edge $e \in E(G)$, use a breadth first search strategy to find all induced cycles C containing edge e such that $|V(C)| \leq z$. Let C_e be the set of all such cycles containing e and let $F = \bigcup_{e\in E(G)} C_e$; // $O(v^3)$
5. For any two different cycles C_1 and C_2 in F, we do // $O(v^6)$
   BEGIN

   (5A) If $V(C_1) \cap V(C_2) = \emptyset$ and there is no edge $e = (v_1, v_2)$, where $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$, then we can construct a new graph G' by contracting C_1 into a vertex x, C_2 into a vertex y, and deleting all parallel edges produced; // $O(v)$

   (5B) We again constructed a new graph G_1 from G', Fig. 1 and Fig. 2 show an example of construction from G' to G_1 (the construction of 4-regular graphs is the same as it). // $O(v^2)$

   (a) Each vertex v in G' becomes two vertices v' and v'' in G_1 and there is an arc $v'v''$ in G_1 from the vertex v' to v'' with arc capacity of 1;

   (b) For each edge of G', we have that: suppose there is an edge $e = uv$ in G', then there are two arcs $e' = u''v'$ and $e'' = v''u'$ in G_1 corresponding to it, and arc capacity of e' and e'' are both ∞.

   (5C) Use the algorithm in Even (2011) (5.3 The Dinitz Algorithm)[1] to find a minimum edge cutset which separates x'' and y' in G_1 (vetex x in G' becomes x' and x'' in G_1, and vertex y in G' becomes y' and y'' in G_1). Then the minimum edge cutset corresponds to a minimum vertex cutset S_{xy} in G' which separates x and y. Note that S_{xy} is also the minimum cyclic vertex cutset separating C_1 and C_2 in G; // $O(|E|v^{1/2}) = O(v^{3/2})$

   (5D) $s := \min\{s, |S_{xy}|\}$; // $O(1)$

   END;
6. Then $c\kappa(G) = s$ and is returned;

Next we analyze the time complexity of Algorithm 1. Since G is a 4-regular graph, $|E| = 2v$, i.e., $O(|E|) = O(v)$. In Step 1, finding the length of the shortest cycle

[1] In Dinitz (2006), Yefim Dinitz tells the differences between his version and Even's Version.

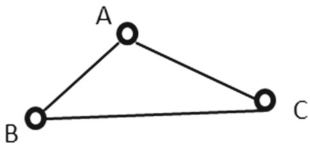


Fig. 1 An example for original graph

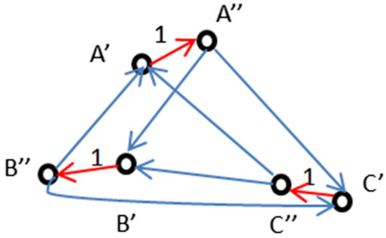


Fig. 2 An example of constructed graph for Fig. 1

containing a vertex v takes $O(|E|)$, so in total $O(|E|v) = O(v^2)$ for all vertices in graph G. In Step 4 [see Lou and Wang (2005), Theorem 4], for each edge $e \in E(G)$, there are at most $O(v^2)$ induced cycles of length at most $4\log_3(2v) + x_0$ (x_0 is a positive integer.) containing e. Hence, there are at most $O(v^2|E|) = O(v^3)$ such cycles in F for all edges in $E(G)$. In Step 5, the **FOR** loop repeats $O(v^6)$ times and Step 5C takes $O(v^{1/2}|E|) = O(v^{3/2})$ (Even and Tarjan 1975, Theorem 3). So Steps 5 including $5A$, $5B$, $5C$ and $5D$ totally take $O(v^{6+3/2}) = O(v^{15/2})$.

Hence Algorithm 1 is an $O(v^{15/2})$ algorithm.

3 The preparation for proving the correctness of Algorithm 1

In this section, we present several lemmas and new terms which will help to prove the correctness of Algorithm 1.

Let G be a 4-regular graph with a cyclic vertex cutset S, and D_1 and D_2 be two components of $G - S$, which have the minimum cycles C_1 and C_2 respectively. Let $c = |V(C_1)|$ be the length of cycle C_1 and $C_1 = a_0a_1 \cdots a_{c-1}a_0$.

In the cyclic vertex cutset S, we define two types of vertices. For each vertex v of the first type, v is adjacent to exactly two different vertices on cycle C_1. For each vertex u of the second type, u is adjacent to three different vertices on C_1, and not adjacent to other vertices of component D_1. For example, in Fig. 3, the vertex v_1 is of the first type and v_2 is of the second type. Furthermore, in all Figures of this paper, the circles filled with black represent the vertices in S.

Notation 3.1 Let S_m be the set of all the vertices of first type, and S_n consists of all the second type of vertices.

Let $G\big[V(D_1) \cup S\big]$ be an induced subgraph by $V(D_1)$ and S, and E_s be the set of edges whose both ends are in S. Let $D_s = G\big[V(D_1) \cup S\big] - E_s - E(C_1)$, which is also a subgraph of G. Note that D_s may be disconnected.

Notation 3.2 $N_0(a_i) = \{a_i\}$, $N_1(a_i) = \{u \mid ua_i \in E(D_s)\}$, and $N_r(a_i) = \{u \mid \exists u_1 \in N_{r-1}(a_i), u \notin \bigcup_{j=0}^{r-1} N_j(a_i),\ u_1 \notin S, uu_1 \in E(D_s)\}$.

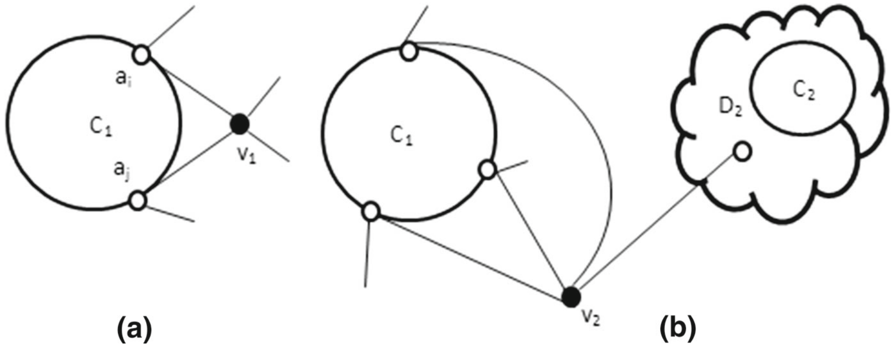
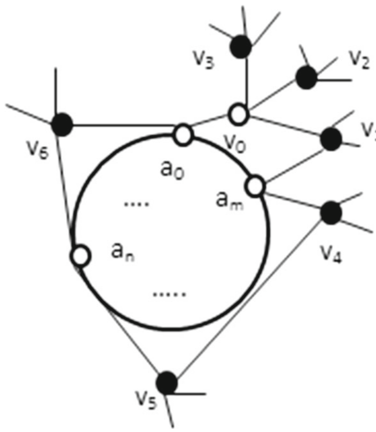


Fig. 3 Two types of vertices in S

Fig. 4 An example for $N_r(a_i)$



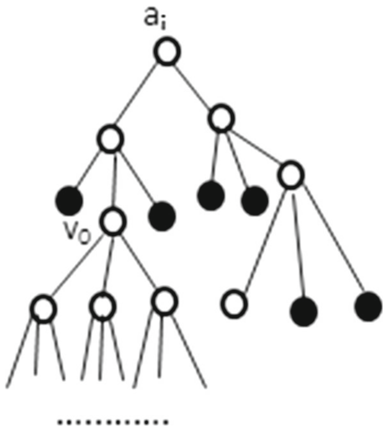
Notation $N_r(a_i)$ $(0 \leq r \leq c/4 - 1)$ denotes the set of vertices, to which the distance are r from a_i in D_s, but not through vertices in $V(C_1) \cup S$. For example, in Fig. 4, the solid vertices belong to cyclic vertex cutset S and $N_1(a_0) = \{v_0, v_6\}$, $N_2(a_0) = \{v_1, v_2, v_3\}$, $N_3(a_0) = \emptyset$, $N_1(a_m) = \{v_1, v_4\}$, $N_2(a_m) = \emptyset$, $N_1(a_n) = \{v_5, v_6\}$, and $N_2(a_n) = \emptyset$.

Note that if $N_{r_1}(a_i) \cap N_{r_2}(a_j) \neq \emptyset$ $(0 \leq i < j \leq c - 1,\ 0 < r_1, r_2 \leq c/4 - 1)$, then all vertices in $N_{r_1}(a_i) \cap N_{r_2}(a_j)$ are in cyclic vertex cutset S. Suppose $v_0 \in N_{r_1}(a_i) \cap N_{r_2}(a_j)$ is not in S. Then $v_0 \in V(D_1)$, and the cycles $a_i\overrightarrow{C_1}a_j \cdots v_0 \cdots a_i$ and $a_j\overrightarrow{C_1}a_i \cdots v_0 \cdots a_j$ are in D_1. Note that $d_{C_1}(a_i, a_j) \leq c/2$. And the distance from a_i (or a_j) to v_0 in D_s is at most $c/4-1$. Hence the length of cycle $a_i\overrightarrow{C_1}a_j \cdots v_0 \cdots a_i$ or $a_j\overrightarrow{C_1}a_i \cdots v_0 \cdots a_j$ in D_1 is at most $c-2 = c/2+2(c/4-1)$, which is a contradiction to the assumption that C_1 is a minimum cycle of D_1.

For any $v_0 \in N_{r_1}(a_i)$, $v_1 \in N_{r_2}(a_i)$ $(r_1 \leq r_2)$ and $v_0 \neq v_1$, if $v_2 \in N_{r_1+1}(a_i) \cap S$, $r_1 < r_2$, and $v_1v_2, v_2v_0 \in E(G)$, then we put the edge v_1v_2 into an edge set E_{T_i}. If $r_1 = r_2$, for $v_1v_2, v_0v_2 \in E(G)$, we only put one of v_1v_2 and v_0v_2 into E_{T_i}.

Notation 3.3 $T(a_i) = G\big[\bigcup_{r=0}^{c/4-1} N_r(a_i)\big] - E_s - E_{T_i}$ $(0 \leq i \leq c - 1)$.

Fig. 5 A T-tree $T(a_i)$



Obviously, $T(a_i)$ is a subgraph of G, and is a tree we call T*-tree*. We say that a T-tree $T(a_i)$ is rooted at a vertex a_i. For example, in Fig. 4, $T(a_0)$ is a tree rooted at a vertex a_0 with non-leaf v_0 and leaves v_1, v_2, v_3, and v_6.

A *full tree* of depth d is a tree rooted at a vertex v' with levels $0, 1, \ldots, d$ such that the vertex v' has three children and each vertex at the levels from 1 to $d-1$ also has three children. We call it a trivial full tree when $d = 0$.

Definition 3.4 Suppose the depth of a T-tree T is d ($d \leq c/4 - 1$). Then the T-tree T contains x subtrees at the kth level if T has x vertices at the kth level ($1 \leq k \leq d$).

The depth of a subtree at the kth level of T is at most $d - k$. Moreover, the subtree is rooted at a vertex v_0 at the kth level of T such that the vertex v_0 and each vertex at the levels from 1 to $d - k - 1$ have the same children as the vertices in T. Note that if there exists one subtree at the kth ($1 \leq k \leq d$) level of T not containing any vertex of cyclic vertex cutset S and the vertices at the levels from 0 to $d - k - 1$ of the subtree are not adjacent to the vertices of S, then the subtree is a full tree. Then the depth of T must be $c/4 - 1$ and that of the subtree is $c/4 - 1 - k$. For example, in Fig. 5, a T-tree $T(a_i)$ contains two subtrees at the 1st level, and six subtrees at the 2nd level, and six subtrees at the 3rd level, and the subtree at the 2nd level rooted at v_0 is a full tree of depth $c/4 - 3$.

Lemma 3.1 *Let G be a 4-regular graph with girth $g \geq 7$. Suppose the cardinality of cyclic vertex cutset $|S| \leq 2g - 1$ and the ranges of length c of a minimum cycle C_1 in component D_1 of $G - S$ are $g < c < 2(g - 4)$ and $c \geq 16$. Then D_1 contains at least $(3^{c/4-1} + 1)/2$ vertices.*

Proof Let $c = |V(C_1)|$ and $C_1 = a_0 a_1 \cdots a_{c-1} a_0$. According to Lemma 1.3, we have $|\bigcup_{i=0}^{c-1} N_1(a_i)| \geq 2g$. Let V_t denote the vertices set of some type belonging to S. Each vertex in this set is adjacent to at least one vertex of C_1 and to at least one vertex of $D_1 - V(C_1)$.

Case (1) Suppose that there is not any edge whose one end belongs to $V(C_1)$, the other end belongs to V_t. Let $x = |\bigcup_{i=0}^{c-1} N_1(a_i)| - |\bigcup_{i=0}^{c-1} N_1(a_i) \cap S|$. Then we have

x vertices in $\bigcup_{i=0}^{c-1} N_1(a_i)$ not belonging to S and at least x subtrees at the 1st level of all T-trees, implying that there are at least $3x$ subtrees at the 2nd level of all T-trees. Note that the $3x$ subtrees at the 2nd level contain $|S| - |\bigcup_{i=0}^{c-1} N_1(a_i) \cap S|$ vertices in the cyclic vertex cutset S. Since $|\bigcup_{i=0}^{c-1} N_1(a_i)| \geq 2g$ and $|S| \leq 2g-1$, we have

$$|S| - \left|\bigcup_{i=0}^{c-1} N_1(a_i) \cap S\right| = |S| - \left(|\bigcup_{i=0}^{c-1} N_1(a_i)| - x\right) \leq x - 1.$$

So the $3x$ subtrees at the 2nd level of all T-trees contain at most $x-1$ vertices in S. However, each vertex in S is contained in at most three subtrees since G is a 4-regular graph. Hence at most $3(x-1)$ subtrees at the 2nd level of all T-trees contain vertices in S and at least $3x - 3(x-1) = 3$ subtrees at the 2nd level of all T-trees do not contain any vertex in S, being full trees of depth $c/4 - 3$. Then we have

$$v(D_1) \geq 3 \cdot (3^0 + 3^1 + \cdots + 3^{c/4-3}) + c \geq (3^{c/4-1} + 1)/2.$$

Case (2) Suppose that there exists an edge whose one end belongs to $V(C_1)$, the other end belongs to V_t.

Then a vertex v_0 must exist such that $v_0 \in T(a_i) \cap T(a_j)$ $(i \neq j)$, $v_0 \in N_1(a_i)$ and $v_0 \notin N_1(a_j)$. Then the distance on C_1 between a_i and a_j is at most $c/2$, and the distance on T-tree $T(a_j)$ between the v_0 and a_j is at most $c/4 - 1$. So, the length of cycle $a_i\overrightarrow{C_1}a_j \cdots v_0 a_i$ or $a_i\overleftarrow{C_1}a_j \cdots v_0 a_i$ is at most $c/2 + (c/4 - 1) + 1 = 3c/4$. Hence, we have

$$3c/4 \geq g. \tag{3.1}$$

According to the inequality (3.1) and $g < c < 2(g-4)$, we have

$$4g/3 \leq c < 2(g-4). \tag{3.2}$$

By Notation 3.1, S_m consists of the vertices in S being adjacent to exactly two different vertices on cycle C_1. Note that $S_m = \emptyset$. Suppose that $v_0 \in S_m$ and $v_0 \in N_1(a_i) \cap N_1(a_j)$. Then we have a cycle of length at most $c/2+2$. Hence, $c/2+2 \geq g$, i.e., $c \geq 2(g-2)$, a contradiction to the inequality (3.2). Furthermore, obviously we have $S_n = \emptyset$.

Note that there is not any edge whose both ends belong to the vertex set $\bigcup_{i=0}^{c-1} N_1(a_i)$. Suppose such an edge exists, then we have a cycle of length at most $c/2 + 3$. Hence, $c/2 + 3 \geq g$, i.e., $c \geq 2(g-3)$, a contradiction to (3.2). Note that a T-tree $T(a_i)$ contains 6 subtrees at the 2nd level if there is no vertex in $N_1(a_i)$ belonging to the cyclic vertex cutset S since $c \geq g+1$, $\bigcup_{i=0}^{c-1} N_1(a_i) \geq 2g+2$ and $|S| \leq 2g-1$.

Note that $N_2(a_i) \cap N_2(a_j) = \emptyset$ for any $a_i, a_j \in V(C_1)$ $(0 \leq i < j \leq c-1)$. Suppose $v_0 \in N_2(a_i) \cap N_2(a_j)$, $v_1 \in N_1(a_i)$ and $v_2 \in N_1(a_j)$ $(0 \leq i < j \leq c-1)$, then the length of two cycles $a_i\overrightarrow{C_1}a_j v_2 v_0 v_1 a_i$ and $a_j\overrightarrow{C_1}a_i v_1 v_0 v_2 a_j$ are greater than or equal to g, thus we have $c \geq 2(g-4)$, a contradiction. In addition, suppose that

$v_3, v_4 \in N_1(a_i)$, $v_5 \in N_2(a_i)$ and $v_3v_5, v_4v_5 \in E(G)$, then we have a cycle of length 4, a contradiction.

Based on three paragraphs discussed above, we next prove that at least 25 subtrees at the 3rd level of all T-trees do not contain any vertex of S and these subtrees are full trees of depth $c/4 - 4$.

Suppose there are x_1 $(0 \leq x_1 \leq |S|)$ vertices in $\bigcup_{i=0}^{c-1} N_1(a_i)$ belonging to S. Since $S_m = \emptyset$ and $S_n = \emptyset$, $2c - x_1$ vertices in $\bigcup_{i=0}^{c-1} N_1(a_i)$ do not belong to S. According to the discussion above, since there is not any edge whose both ends belong to $\bigcup_{i=0}^{c-1} N_1(a_i)$ and $N_2(a_i) \cap N_2(a_j) = \emptyset$ for any $a_i, a_j \in V(C_1)$ $(0 \leq i < j \leq c-1)$, then there are $3(2c - x_1)$ vertices in $\bigcup_{i=0}^{c-1} N_2(a_i)$. Let P_1 be a set of these $3(2c - x_1)$ vertices, i.e., $|P_1| = 3(2c - x_1)$. Suppose there are x_2 $(0 \leq x_2 \leq |S| - x_1)$ vertices in $\bigcup_{i=0}^{c-1} N_2(a_i)$ belonging to S. Let P_2 be a set of $3(2c - x_1) - x_2$ vertices in $\bigcup_{i=0}^{c-1} N_2(a_i)$ not belonging to S, i.e., $|P_2| = 3(2c - x_1) - x_2$.

Each vertex in $\bigcup_{i=0}^{c-1} N_1(a_i) \cap S$ consisting of x_1 vertices may be adjacent to at most two vertices in P_2. Each vertex in $P_1 - P_2$ consisting of x_2 vertices may be adjacent to at most two vertices in P_2. Each vertex in $S - \bigcup_{i=0}^{c-1} N_1(a_i) \cap S - (P_1 - P_2)$ consisting of $|S| - x_1 - x_2$ vertices may be adjacent to at most three vertices in P_2 since G is a 4-regular graph.

From the paragraph above, we can see that there are at most $2x_1$ edges between P_2 and $\bigcup_{i=0}^{c-1} N_1(a_i) \cap S$, at most $2x_2$ edges between P_2 and $P_1 - P_2$, and at most $3(|S| - x_1 - x_2)$ edges between P_2 and $S - \bigcup_{i=0}^{c-1} N_1(a_i) \cap S - (P_1 - P_2)$. However, there are $3|P_2|$ edges which are incident with the vertices in P_2, except for those edges between P_2 and $\bigcup_{i=0}^{c-1} N_1(a_i) - S$. Hence, there are at least $3|P_2| - 2x_1 - 2x_2 - 3(|S| - x_1 - x_2) \geq 25$ [the detailed computation is given in (3.3)] edges which are not incident with the vertices of S, implying that at least 25 subtrees at the 3rd level in $\bigcup_{i=0}^{c-1} T(a_i)$ do not contain any vertex of S. Then the 25 subtrees are full trees of depth $c/4 - 4$. Hence, we have $v(D_1) \geq 25 \cdot (3^0 + 3^1 + \cdots + 3^{c/4-4}) + c \geq (3^{c/4-1} + 1)/2$ since $c \geq 16$.

Since $0 \leq x_1 \leq |S|$, $0 \leq x_2 \leq |S| - x_1$, $4g/3 \leq c < 2(g - 4)$, $|S| \leq 2g - 1$ and $g \geq 7$, we have

$$\begin{aligned} &3|P_2| - 2x_1 - 2x_2 - 3(|S| - x_1 - x_2) \\ &\quad \geq 3[3(2c - x_1) - x_2] - 2x_1 - 2x_2 - 3(|S| - x_1 - x_2) \\ &\quad = 18c - 3|S| - 2(x_2 + 4x_1) \\ &\quad \geq 2g + 11 \geq 25. \end{aligned} \tag{3.3}$$

□

Lemma 3.2 *Let $3g - 8 \leq c \leq 3g - 2$ and $g \geq 13$. Suppose there exists a vertex a_i on C_1 such that each vertex in $N_1(a_i)$ belongs to S_m, then there is no vertex in $N_1(a_{i-1}) \cup N_1(a_{i+1})$ belonging to S_m.*

Proof Recall that S_m consists of the vertices in S being adjacent to exactly two different vertices on cycle C_1, and $d_{C_1}(a_i, a_j)$ denotes the distance between vertex a_i and vertex a_j on cycle C_1.

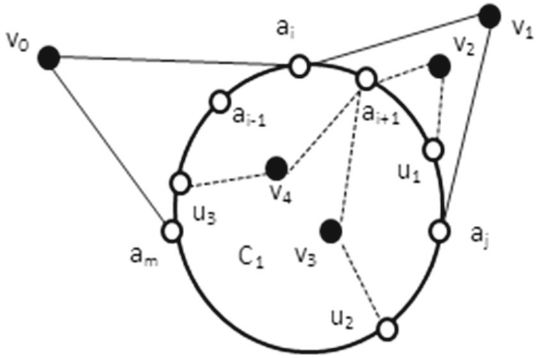


Fig. 6 $3g-8 \le c \le 3g-2$

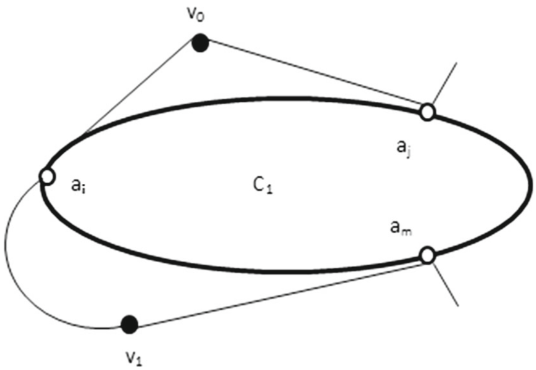


Fig. 7 $c \ge 3g-8$

We firstly prove that there is no vertex in $N_1(a_{i+1})$ belonging to S_m. As in Fig. 6, suppose the vertex $v_0 \in N_1(a_i) \cap N_1(a_m)$ and vertex $v_1 \in N_1(a_i) \cap N_1(a_j)$. Since the lengths of three cycles $a_i\overrightarrow{C_1}a_jv_1a_i$, $a_iv_1a_j\overrightarrow{C_1}a_mv_0a_i$ and $a_m\overrightarrow{C_1}a_iv_0a_m$ are greater than or equal to g, we have that $d_{C_1}(a_i,a_j) \ge g-2$, $d_{C_1}(a_i,a_m) \ge g-2$, and $d_{C_1}(a_m,a_j) \ge g-4$. Obviously, the three inequalities take equal sign when $c = 3g-8$. Moreover, $d_{C_1}(a_i,a_j) \le g+4$, $d_{C_1}(a_i,a_m) \le g+4$ and $d_{C_1}(a_m,a_j) \le g+2$ when $3g-8 \le c \le 3g-2$.

Case (1) Suppose $u_1 \in C_1^+[a_i,a_j]$ and $v_2 \in N_1(a_{i+1}) \cap N_1(u_1)$. If $u_1 \in C_1^+[a_{i+2},a_{j-6}]$, then $d_{C_1}(u_1,a_j) \ge 6$ and $d_{C_1}(a_{i+1},u_1) = d_{C_1}(a_i,a_j) - d_{C_1}(a_i,a_{i+1}) - d_{C_1}(u_1,a_j) \le g+4-7 = g-3$. Thus the length of cycle $a_{i+1}\overrightarrow{C_1}u_1v_2a_{i+1}$ is less than or equal to $g-1$, a contradiction; If $u_1 \in C_1^+[a_{j-5},a_j]$, then the length of cycle $a_iv_1a_j\overleftarrow{C_1}u_1v_2a_{i+1}a_i$ is at most $10 < g$, a contradiction; If $u_1 \in C_1^+[a_i,a_{i+1}]$, obviously we have a cycle of length less than g, a contradiction.

Case (2) Suppose $u_2 \in C_1^+(a_j,a_m)$ and $v_3 \in N_1(a_{i+1}) \cap N_1(u_2)$. We have $d_{C_1}(a_j,u_2) \le (g+2)/2$ or $d_{C_1}(u_2,a_m) \le (g+2)/2$ since $d_{C_1}(a_m,a_j) \le g+2$. However, $d_{C_1}(a_j,u_2) \ge g-5$ and $d_{C_1}(u_2,a_m) \ge g-5$ since the lengths of cycles $a_ia_{i+1}v_3u_2\overrightarrow{C_1}a_mv_0a_i$ and $a_iv_1a_j\overrightarrow{C_1}u_2v_3a_{i+1}a_i$ are greater than or equal to g. Hence, we have that $(g+2)/2 \ge g-5$, i.e., $g \le 12$, which is a contradiction to the assumption that $g \ge 13$.

Case (3) Suppose $u_3 \in C_1^+[a_m, a_i)$ and $v_4 \in N_1(a_{i+1}) \cap N_1(u_3)$. Then $d_{C_1}(u_3, a_i) \geq g - 3$ since the length of cycle $u_3\overrightarrow{C_1}a_i a_{i+1} v_4 u_3$ is greater than or equal to g. Then we have $d_{C_1}(a_m, u_3) = d_{C_1}(a_m, a_i) - d_{C_1}(u_3, a_i) \leq 7$ since $d_{C_1}(a_i, a_m) \leq g + 4$. The length of cycle $a_m\overrightarrow{C_1}u_3 v_4 a_{i+1} a_i v_0 a_m$ is less than or equal to 12, which is a contradiction to the assumption that $g \geq 13$.

We know that there is no vertex in $N_1(a_{i+1})$ belonging to S_m from the above discussion. The discussion of $N_1(a_{i-1})$ is similar to that of $N_1(a_{i+1})$. Then there is no vertex in $N_1(a_{i-1})$ belonging to S_m. Hence the lemma is proved. □

Lemma 3.3 *Let* $g \geq 13$. *If* $2(g-2) \leq c \leq 3g-2$, *then* $|S_m| \leq c/2$.

Proof **Case (1)** Suppose that $2(g-2) \leq c \leq 3g-9$.

Then we have that the cardinality of each $N_1(a_i)(0 \leq i \leq c-1)$ is two since G is a 4-regular graph. Suppose there exists a vertex a_i such that the two vertices in $N_1(a_i)$ both belong to S_m. As in Fig. 7, assuming that a vertex $v_0 \in N_1(a_i) \cap N_1(a_j)$ and a vertex $v_1 \in N_1(a_i) \cap N_1(a_m)$ $(i < j < m)$. Then the lengths of three cycles $a_i\overrightarrow{C_1}a_j v_0 a_i$, $a_i\overleftarrow{C_1}a_m v_1 a_i$ and $a_i v_0 a_j \overrightarrow{C_1} a_m v_1 a_i$ are greater than or equal to g. So we have that $d_{C_1}(a_i, a_j) \geq g-2$, $d_{C_1}(a_i, a_m) \geq g-2$ and $d_{C_1}(a_j, a_m) \geq g-4$. Hence, we have $c \geq 2(g-2) + g - 4 \geq 3g - 8$, which is a contradiction to the assumption that $2(g-2) \leq c \leq 3g-9$.

So for any vertex a_i on C_1, at most one vertex in $N_1(a_i)$ belongs to S_m. Since the cycle C_1 has c vertices, at most c edges are between $V(C_1)$ and S_m. Since each vertex in S_m is adjacent to two vertices on C_1, we can infer that $|S_m| \leq c/2$.

Case (2) Suppose that $3g-8 \leq c \leq 3g-2$. Then the discussion is similar to that of $2(g-2) \leq c \leq 3g-9$ if $S_m = \emptyset$. If $S_m \neq \emptyset$, then according to Lemma 3.2, we have that if there exists a vertex a_i on C_1 such that each vertex in $N_1(a_i)$ belongs to S_m $(0 \leq i \leq c-1)$, then there is no vertex in $N_1(a_{i-1}) \cup N_1(a_{i+1})$ belonging to S_m, implying that at most c edges are between $V(C_1)$ and S_m. Since S_m consists of the vertices in S being adjacent to exactly two different vertices on C_1, we can infer that $|S_m| \leq c/2$. □

Lemma 3.4 *Let* $g \geq 13$. *If* $2(g-2) \leq c \leq 3g-2$, *then* $|S_n| \leq 2$.

Proof By Notation 3.1, S_n consists of the vertices in S being adjacent to three different vertices on cycle C_1.

Case (1) Suppose that $2(g-2) \leq c < 3g-6$. Obviously, we have that $|S_n| = 0$. Suppose that $|S_n| \neq 0$. As in Fig. 8a, if the vertex v_0 is adjacent to the vertices u_1, u_3 and u_6 on cycle C_1, then we have that $d_{C_1}(u_1, u_3) \geq g-2$, $d_{C_1}(u_3, u_6) \geq g-2$ and $d_{C_1}(u_6, u_1) \geq g-2$ since the lengths of three cycles $v_0 u_1 \overrightarrow{C_1} u_3 v_0$, $v_0 u_3 \overrightarrow{C_1} u_6 v_0$, and $v_0 u_6 \overrightarrow{C_1} u_1 v_0$ are greater than or equal to g. Hence, $c = d_{C_1}(u_1, u_3) + d_{C_1}(u_3, u_6) + d_{C_1}(u_6, u_1) \geq 3g-6$, which is a contradiction to the assumption that $2(g-2) \leq c < 3g-6$.

Case (2) Suppose that $3g-6 \leq c \leq 3g-2$.

Let the vertices $v_0, v_1 \in S_n$. Suppose the vertex v_0 is adjacent to the vertices u_1, u_3 and u_6 on the cycle C_1, and v_1 is adjacent to the vertices u_2, u_4 and u_5 on C_1. Suppose

$u_2 \in C_1^+(u_1, u_3)$ and $u_4 \in C_1^+(u_3, u_6)$, then $u_5 \in C_1^+(u_6, u_1)$. Otherwise, if $u_5 \in C_1^+(u_4, u_6)$ as in Fig. 8a, then the lengths of three cycles $u_2\overrightarrow{C_1}u_4v_1u_2$, $v_1u_4\overrightarrow{C_1}u_5v_1$ and $u_6\overrightarrow{C_1}u_1v_0u_6$ are greater than or equal to g, thus we have that $d_{C_1}(u_2, u_4) \geq g-2$, $d_{C_1}(u_4, u_5) \geq g-2$ and $d_{C_1}(u_6, u_1) \geq g-2$. Since $c \leq 3g-2$, then $d_{C_1}(u_1, u_2) + d_{C_1}(u_5, u_6) = c - d_{C_1}(u_2, u_4) - d_{C_1}(u_4, u_5) - d_{C_1}(u_6, u_1) \leq 4$. So we have a cycle $u_1\overrightarrow{C_1}u_2v_1u_5\overrightarrow{C_1}u_6v_0u_1$ of length less than or equal to 8, which is a contradiction to the assumption that $g \geq 13$. Hence, the structure shown in Fig. 8a does not exist. Similarly, we have $u_5 \notin C_1^+(u_1, u_3)$.

Suppose $u_2 = u_1$. Obviously, we have $u_4 \neq u_3$, $u_4 \neq u_6$, $u_5 \neq u_3$ and $u_5 \neq u_6$. Otherwise, it induces a cycle of length at most 4, a contradiction. Furthermore, if $u_4 \in C_1^+(u_3, u_6)$, then the discussion will be similar to that of the above paragraph, hence we have $u_5 \in C_1^+(u_6, u_1)$.

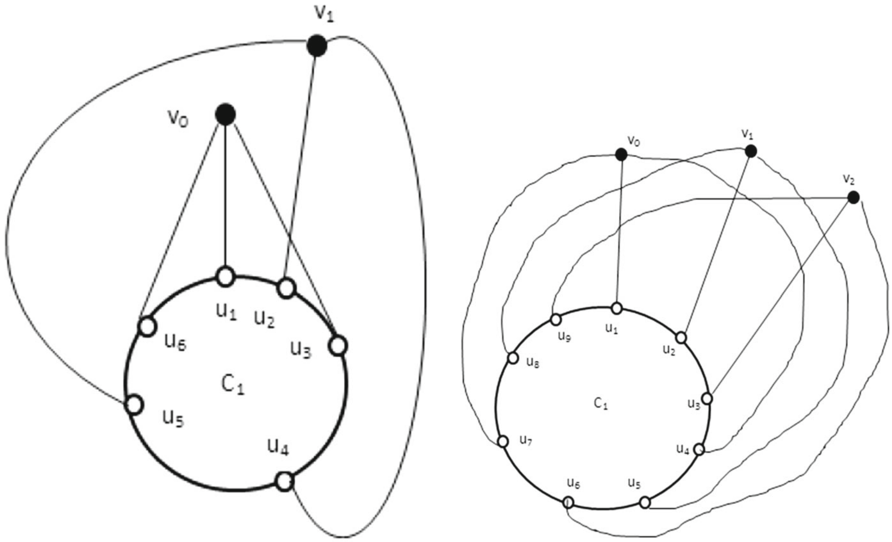
Hence, suppose the vertices $v_0, v_1 \in S_n$, v_0 is adjacent to the vertices u_1, u_3 and u_6 on cycle C_1, and v_1 is adjacent to the vertices u_2, u_4 and u_5 on cycle C_1. Then any two of the vertices u_2, u_4 and u_5 cannot belong to a vertex set at the same time, where the vertex set is $C_1^+(u_1, u_3)$, $C_1^+(u_3, u_6)$, or $C_1^+(u_6, u_1)$. By the same reason, any two of the vertices u_1, u_3 and u_6 cannot belong to the same vertex set of $C_1^+(u_2, u_4)$, $C_1^+(u_4, u_5)$, or $C_1^+(u_5, u_2)$.

Suppose that there are three vertices belonging to S_n, then the structure is shown as Fig. 8b according to the discussion above (intuitively, we suppose any two of these three vertices are not adjacent to the same vertex on C_1, the discussion is basically the same). In Fig. 8b, the three vertices v_0, v_1 and v_2 are contained in S_n. Besides, the vertex v_0 is adjacent to the vertices u_1, u_4 and u_7 on cycle C_1, v_1 is adjacent to the vertices u_2, u_5 and u_8 on C_1, and v_2 is adjacent to the vertices u_3, u_6 and u_9 on C_1. Note that the vertices $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8$ and u_9 are arranged on cycle C_1 in the clockwise direction.

Suppose that $c = 3g-6$, $d_{C_1}(u_1, u_2) = x$ and $d_{C_1}(u_2, u_3) = y$. Then $d_{C_1}(u_1, u_4) = g-2$, $d_{C_1}(u_4, u_7) = g-2$ and $d_{C_1}(u_7, u_1) = g-2$ since the lengths of three cycles $v_0u_1\overrightarrow{C_1}u_4v_0$, $v_0u_4\overrightarrow{C_1}u_7v_0$, and $v_0u_7\overrightarrow{C_1}u_1v_0$ are greater than or equal to g. Similarly, we have that $d_{C_1}(u_2, u_5) = g-2$, $d_{C_1}(u_5, u_8) = g-2$, $d_{C_1}(u_8, u_2) = g-2$, $d_{C_1}(u_3, u_6) = g-2$, $d_{C_1}(u_6, u_9) = g-2$ and $d_{C_1}(u_9, u_3) = g-2$. Hence $d_{C_1}(u_3, u_4) = d_{C_1}(u_1, u_4) - d_{C_1}(u_1, u_2) - d_{C_1}(u_2, u_3) = g-2-x-y$. Similarly, we have that $d_{C_1}(u_4, u_5) = x$, $d_{C_1}(u_5, u_6) = y$, $d_{C_1}(u_6, u_7) = g-2-x-y$, $d_{C_1}(u_7, u_8) = x$, $d_{C_1}(u_8, u_9) = y$, $d_{C_1}(u_9, u_1) = g-2-x-y$. Then the length $2x+4$ of cycle $v_0u_1\overrightarrow{C_1}u_2v_1u_8\overleftarrow{C_1}u_7v_0$ is greater than or equal to g, i.e., $2x+4 \geq g$. Similarly, we have that $2y+4 \geq g$ and $2(g-2-x-y)+4 \geq g$ since the lengths of two cycles $v_1u_2\overrightarrow{C_1}u_3v_2u_9\overleftarrow{C_1}u_8v_1$ and $v_2u_3\overrightarrow{C_1}u_4v_0u_1\overleftarrow{C_1}u_9v_2$ are greater than or equal to g.

Let $0 \leq t_1+t_2+t_3 \leq 4$ ($t_1, t_2, t_3 \geq 0$). When $3g-6 \leq c \leq 3g-2$, we have that

$$\begin{cases} 2x+4+t_1 \geq g, \\ 2y+4+t_2 \geq g, \\ 2(g-2-x-y)+4+t_3 \geq g. \end{cases} \tag{3.4}$$



(a) A structure **(b)** Another structure

Fig. 8 Two structures that do not exist for $c \leq 3g-2$ and $g \geq 13$

We have following inequality from (3.4),

$$(g-4-t_1)/2+(g-4-t_2)/2+(g-4-t_3)/2 \leq g-2. \tag{3.5}$$

Hence, we have that $3(g-4)/2 \leq (t_1+t_2+t_3)/2+g-2$. Since $0 \leq t_1+t_2+t_3 \leq 4$, so $3(g-4)/2 \leq g$, i.e., $g \leq 12$, which is a contradiction to the assumption that $g \geq 13$. Therefore there are not three vertices belonging to S_n and $|S_n| \leq 2$. □

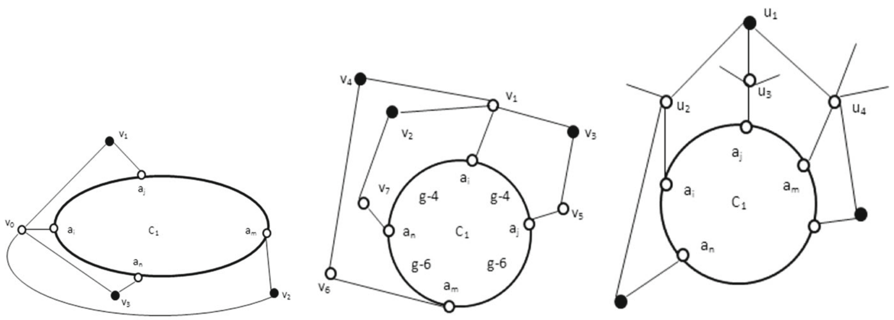
Lemma 3.5 *Let G be a 4-regular graph with girth $g \geq 19$. Suppose the cardinality of cyclic vertex cutset $|S| \leq 2g-1$ and the range of length c of a minimum cycle C_1 in a component D_1 of $G-S$ is $2(g-4) \leq c \leq 3g-2$. Then D_1 contains at least $(3^{c/4-1}+1)/2$ vertices.*

Proof **Observation 1** We shall prove that there is no vertex $v_0 \in N_1(a_i)$ such that the vertex v_0 is adjacent to the vertices $v_1 \in N_1(a_j)$, $v_2 \in N_1(a_m)$ and $v_3 \in N_1(a_n)$ $(0 \leq i \leq j \leq m \leq n \leq c-1)$.

Suppose that the equality $i=j$ holds, then obviously there is no vertex $v_0 \in N_1(a_i)$ such that the vertex v_0 is adjacent to the vertex $v_1 \in N_1(a_j)$. Otherwise, it yields a cycle $v_0v_1a_iv_0$ of length 3, a contradiction.

Suppose that the equality $j=m$ holds, then obviously there is no vertex $v_0 \in N_1(a_i)$ such that the vertex v_0 is adjacent to the vertices $v_1 \in N_1(a_j)$, $v_2 \in N_1(a_m)$ and $v_3 \in N_1(a_n)$. Otherwise, it yields a cycle $v_0v_1a_jv_2v_0$ of length 4, a contradiction. If $j=n$ or $m=n$, then the discussion is similar to that of $j=m$.

Suppose that for any i, j, m, n $(0 \leq i < j < m < n \leq c-1)$, there exists a vertex v_0 in $N_1(a_i)$ such that the vertex v_0 is adjacent to the vertices $v_1 \in N_1(a_j)$, $v_2 \in N_1(a_m)$ and $v_3 \in N_1(a_n)$. As in Fig. 9a, then the lengths of four cycles $v_0a_i\overrightarrow{C_1}a_jv_1v_0$,



(a) The first structure **(b)** The second structure **(c)** The third structure

Fig. 9 Three structures that do not exist for $c \leq 3g-2$ and $g \geq 19$

$v_0 a_i \overleftarrow{C_1} a_n v_3 v_0$, $v_0 v_1 a_j \overrightarrow{C_1} a_m v_2 v_0$ and $v_0 v_2 a_m \overrightarrow{C_1} a_n v_3 v_0$ are greater than or equal to g. So $d_{C_1}(a_i, a_j) \geq g-3$, $d_{C_1}(a_j, a_m) \geq g-4$, $d_{C_1}(a_m, a_n) \geq g-4$ and $d_{C_1}(a_n, a_i) \geq g-3$. Hence, we have

$$c = d_{C_1}(a_i, a_j) + d_{C_1}(a_j, a_m) + d_{C_1}(a_m, a_n) + d_{C_1}(a_n, a_i) \geq 4g - 14. \quad (3.6)$$

However, since $c \leq 3g-2$, we have that $4g-14 \leq 3g-2$, i.e., $g \leq 12$, which is a contradiction to the assumption that $g \geq 19$. Hence, for any $a_i, a_j, a_m, a_n \in V(C_1)$ $(0 \leq i \leq j \leq m \leq n \leq c-1)$, there is no vertex $v_0 \in N_1(a_i)$ such that the vertex v_0 is adjacent to the vertices $v_1 \in N_1(a_j)$, $v_2 \in N_1(a_m)$ and $v_3 \in N_1(a_n)$.

Observation 2 Assume that there are vertex $a_i \in V(C_1)$ and vertex $v_1 \in N_1(a_i)$ such that $a_i v_1 \in E(G)$. We shall prove that the vertex v_1 cannot be adjacent to a vertex $v_2 \in N_2(a_i) \cap N_2(a_n)$, a vertex $v_3 \in N_2(a_i) \cap N_2(a_j)$ and a vertex $v_4 \in N_2(a_i) \cap N_2(a_m)$ at the same time ($i \neq j, m$ and n).

Suppose that there are a vertex $a_i \in V(C_1)$ and a vertex $v_1 \in N_1(a_i)$ such that $a_i v_1 \in E(G)$, and the vertex v_1 is adjacent to a vertex $v_2 \in N_2(a_i) \cap N_2(a_n)$, a vertex $v_3 \in N_2(a_i) \cap N_2(a_j)$ and a vertex $v_4 \in N_2(a_i) \cap N_2(a_m)$ at the same time ($i \neq j, m$ and n).

When $j = m$, or $m = n$, or $j = n$ holds, clearly it yields a cycle of length at most 6, a contradiction.

Next assume $j < m < n$ holds. As in Fig. 9b, assume that vertices $v_1 \in N_1(a_i)$, $v_5 \in N_1(a_j)$, $v_6 \in N_1(a_m)$, $v_7 \in N_1(a_n)$, $v_2 \in N_2(a_i) \cap N_2(a_n)$, $v_3 \in N_2(a_i) \cap N_2(a_j)$, and $v_4 \in N_2(a_i) \cap N_2(a_m)$. Moreover, the edges v_1v_3, v_3v_5, v_1v_2, v_2v_7, v_1v_4 and v_4v_6 are in $E(G)$. Then the lengths of four cycles $a_i v_1 v_3 v_5 a_j \overleftarrow{C_1} a_i$, $a_i v_1 v_2 v_7 a_n \overrightarrow{C_1} a_i$, $v_1 v_3 v_5 a_j \overrightarrow{C_1} a_m v_6 v_4 v_1$ and $v_1 v_4 v_6 a_m \overrightarrow{C_1} a_n v_7 v_2 v_1$ are greater than or equal to g. Then we have $d_{C_1}(a_i, a_j) \geq g-4$, $d_{C_1}(a_j, a_m) \geq g-6$, $d_{C_1}(a_m, a_n) \geq g-6$ and $d_{C_1}(a_n, a_i) \geq g-4$. So,

$$c = d_{C_1}(a_i, a_j) + d_{C_1}(a_j, a_m) + d_{C_1}(a_m, a_n) + d_{C_1}(a_n, a_i) \geq 4g - 20. \quad (3.7)$$

Hence, we have $4g-20 \leq 3g-2$, i.e., $g \leq 18$, a contradiction.

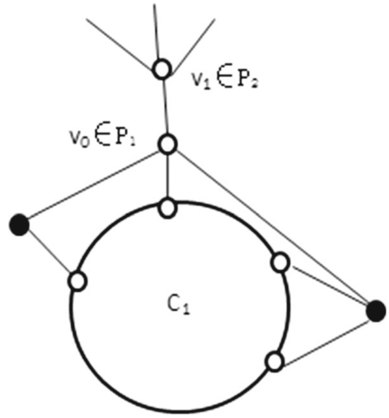


Fig. 10 $2(g-2) \leq c \leq 3g-2$ and $g \geq 19$

Observation 3 Assume that there exists vertex $a_i \in V(C_1)$ and vertex $v_1 \in N_1(a_i)$ such that $a_i v_1 \in E(G)$. In this case, the analysis is similar to Observation 2. Then we can get the result that the vertex v_1 cannot be adjacent to a vertex $v_2 \in N_2(a_i) \cap \left(\bigcup_{r=0}^{2} N_r(a_n)\right)$, a vertex $v_3 \in N_2(a_i) \cap \left(\bigcup_{r=0}^{2} N_r(a_j)\right)$ and a vertex $v_4 \in N_2(a_i) \cap \left(\bigcup_{r=0}^{2} N_r(a_m)\right)$ at the same time ($i \neq j, m$ and n). The reason is that the lower bound of c in the discussion of Observation 3 is larger than the lower bound $4g-20$ in the inequality (3.7) of Observation 2. Hence, we have $4g-20 \leq 3g-2$, i.e., $g \leq 18$, a contradiction.

Observation 4 We shall prove that the component D_1 contains at least 9 subtrees at the 2nd level and the 3rd level of all T-trees, which are full trees of depth at least $c/4-4$.

Note that $S_m = \emptyset$ and $S_n = \emptyset$ when $2(g-4) \leq c < 2(g-2)$. Suppose that $S_m \neq \emptyset$ and a vertex $v_0 \in S_m$. Let $v_0 \in N_1(a_i) \cap N_1(a_j)$. Then we have a cycle of length at most $c/2+2$. Hence, $c/2+2 \geq g$, i.e., $c \geq 2(g-2)$, a contradiction to the inequality $2(g-4) \leq c < 2(g-2)$. Similarly, we have $S_n = \emptyset$.

We assume that $|S_m| = c/2 - x$ $(x \geq 0)$ since $|S_m| \leq c/2$ according to Lemma 3.3 and the discussion above. Let $z = |S_n|$. Then $0 \leq z \leq 2$ according to Lemma 3.4 and the discussion above. Note that the vertices in S_m and S_n belong to $\bigcup_{i=0}^{c-1} N_1(a_i)$ and cyclic vertex cutset S. Suppose that there are y $\Big(0 \leq y \leq |S| - (c/2 - x) - z\Big)$ vertices in $\bigcup_{i=0}^{c-1} N_1(a_i) - S_m - S_n$ belonging to S. Let $P_1 = \left(\bigcup_{i=0}^{c-1} N_1(a_i)\right) - S$. Then we have that $|P_1| = 2c - 2(c/2 - x) - y - 3z$.

Note that at least $|P_1|$ vertices in $\bigcup_{i=0}^{c-1} N_1(a_i)$ do not belong to S. Since $c \geq 2(g-4)$, $g \geq 19$, $x \geq 0$, $0 \leq z \leq 2$ and $|S| \leq 2g-1$, we have that $|P_1| - [|S| - (c/2 - x) - y - z] > 0$, and hence $|P_1| > 0$. As Fig. 10, suppose a vertex $v_0 \in P_1$. Then the vertex v_0 is adjacent to at most two vertices in $\left(\bigcup_{i=0}^{c-1} N_1(a_i)\right) \cap S$ and at least one vertex $v_1 \notin \bigcup_{i=0}^{c-1} \left(N_1(a_i) \cup N_0(a_i)\right)$ according to Observation 1. Then the vertex v_1 either belongs to S, or is adjacent to at most three vertices other than v_0 since G is 4-regular graph.

Suppose that $N_2(a_i) \cap N_2(a_j) = \emptyset$ $(0 \le i, j \le c-1)$. Since each vertex in P_1 is adjacent to at most two vertices in $\left(\bigcup_{i=0}^{c-1} N_1(a_i)\right) \cap S$ and at least one vertex not in $\bigcup_{i=0}^{c-1}\left(N_1(a_i) \cup N_0(a_i)\right)$ according to Observation 1, at least $|P_1|$ vertices not belonging to $\bigcup_{i=0}^{c-1}\left(N_1(a_i) \cup N_0(a_i)\right)$ are adjacent to those vertices in P_1.

Suppose that $N_2(a_i) \cap N_2(a_j) \neq \emptyset$ $(0 \le i, j \le c-1)$. If $u_1, u_2 \in N_1(a_i)$, $u_3 \in N_2(a_i)$ and $u_1u_3, u_2u_3 \in E(G)$, then we have a cycle of length 4, a contradiction. Combining Observation 2 and Observation 3, we see that for each vertex v' in P_1, at least one vertex belonging to $\bigcup_{i=0}^{c-1} N_2(a_i)$ is adjacent to v', but the vertex does not belong to $N_{r_1}(a_i) \cap N_{r_2}(a_j)$ for any $0 \le r_1 \le 2$, $0 \le r_2 \le 2$, i and j.

From the two paragraphs above, we see that for each vertex v' in P_1, there is at least one vertex being adjacent to v' such that the vertex belongs to $\bigcup_{i=0}^{c-1} N_2(a_i)$, but not to $N_{r_1}(a_i) \cap N_{r_2}(a_j)$ for any $0 \le r_1 \le 2$, $0 \le r_2 \le 2$, i and j. So in total there are at least $|P_1| = 2c - 2(c/2 - x) - y - 3z$ vertices being adjacent to those vertices in P_1 and these vertices belong to $\bigcup_{i=0}^{c-1} N_2(a_i)$, but not to $N_{r_1}(a_i) \cap N_{r_2}(a_j)$ for any $0 \le r_1 \le 2$, $0 \le r_2 \le 2$, i and j. Let P_2 be a set of these $|P_1|$ vertices mentioned above, i.e., $|P_2| = |P_1|$. Suppose t $\left(0 \le t \le |S| - (c/2 - x) - y - z\right)$ vertices in P_2 belong to cyclic vertex cutset S. Then Let P_3 be a set of $|P_2| - t$ vertices in P_2 not belonging to S. Note that $|P_3| = |P_2| - t = 2c - 2(c/2 - x) - y - 3z - t$ and $t = |P_2| - |P_3|$.

Each vertex in S_m consisting of $c/2 - x$ vertices may be adjacent to at most one vertex in $P_1 \cup P_3$. Each vertex in $\left(\bigcup_{i=0}^{c-1} N_1(a_i)\right) \cap S - S_m - S_n$ consisting of y vertices may be adjacent to at most two vertices in $P_1 \cup P_3$. Each vertex in $P_2 - P_3$ consisting of t vertices in S may be adjacent to at most three vertices in $P_1 \cup P_3$. Each vertex in $S - S_m - S_n - \left(\left(\bigcup_{i=0}^{c-1} N_1(a_i)\right) \cap S - S_m - S_n\right) - (P_2 - P_3)$ consisting of $|S| - (c/2 - x) - y - z - t$ vertices may be adjacent to at most three vertices in $P_1 \cup P_3$ since G is a 4-regular graph.

From the paragraph above, we can see that there are at most $c/2 - x$ edges between $P_1 \cup P_3$ and S_m, at most $2y$ edges between $P_1 \cup P_3$ and $\left(\bigcup_{i=0}^{c-1} N_1(a_i)\right) \cap S - S_m - S_n$, at most $3t$ edges between $P_1 \cup P_3$ and $P_2 - P_3$, and at most $3(|S| - (c/2 - x) - y - z - t)$ edges between $P_1 \cup P_3$ and $S - S_m - S_n - \left(\left(\bigcup_{i=0}^{c-1} N_1(a_i)\right) \cap S - S_m - S_n\right) - (P_2 - P_3)$. However, there are $2|P_1| + 3|P_3|$ edges which are incident with vertices in $P_1 \cup P_3$, except for those edges between P_1 and $V(C_1)$, and those edges between P_1 and P_3. Hence, there are at least $2|P_1| + 3|P_3| - (c/2 - x) - 2y - 3t - 3[|S| - (c/2 - x) - y - z - t] \ge 8c + 4x - 14g + 7 - 8z$ [the detailed computation is given in (3.8)] edges which are not incident with the vertices of S, implying that at least $8c + 4x - 14g + 7 - 8z$ subtrees at the 2nd level and the 3rd level in $\bigcup_{i=0}^{c-1} T(a_i)$ do not contain any vertex of S. Then the $8c + 4x - 14g + 7 - 8z$ subtrees are full trees of depth at least $c/4 - 4$.

Since $0 \le y \le |S| - (c/2 - x) - z$ and $0 \le t \le |S| - (c/2 - x) - y - z$, we have that $0 \le 4y + 3t \le 4[|S| - (c/2 - x) - z]$. Since $|S| \le 2g - 1$, $|P_1| = 2c - 2(c/2 - x) - y - 3z$ and $|P_3| = 2c - 2(c/2 - x) - y - 3z - t$, we have

$$
\begin{aligned}
&2|P_1| + 3|P_3| - (c/2 - x) - 2y - 3t - 3[|S| - (c/2 - x) - y - z - t] \\
&\quad \geq 10c - 8(c/2 - x) - 3(2g - 1) - (4y + 3t) - 12z \qquad (3.8) \\
&\quad \geq 8c + 4x - 14g + 7 - 8z.
\end{aligned}
$$

If $2(g-4) \leq c < 2(g-2)$, then $|S_m| = 0$ and $|S_n| = 0$, i.e., $x = c/2$ and $z = 0$. If $2(g-2) \leq c \leq 3g-7$, then $z = |S_n| = 0$ and $x \geq 0$. If $3g-6 \leq c \leq 3g-2$, then $z = |S_n| \leq 2$ according to Lemma 3.4 and $x \geq 0$. Hence, we verify that the inequality $8c + 4x - 14g + 7 - 8z \geq 9$ holds when $2(g-4) \leq c \leq 3g-2$.

Therefore at least 9 subtrees at the 2nd level and the 3rd level are full trees of depth at least $c/4 - 4$ and contained in D_1, and $v(D_1) \geq 9 \cdot (3^0 + 3^1 + \cdots + 3^{c/4-4}) + c$. Since $2(g-4) \leq c \leq 3g-2$ and $g \geq 19$, we have $c \geq 30$. Hence, we have $v(D_1) \geq 9 \cdot (3^0 + 3^1 + \cdots + 3^{c/4-4}) + c \geq (3^{c/4-1} + 1)/2$. □

Lemma 3.6 *Let G be a 4-regular graph with girth $g \geq 19$. Suppose the cardinality of cyclic vertex cutset $|S| \leq 2g-1$. Then a component D_1 containing a cycle in $G - S$ contains at least $(3^{c/4-1} + 1)/2$ vertices.*

Proof Since $g \geq 19$, according to Lemma 1.3, $|\bigcup_{i=0}^{c-1} N_1(a_i)| \geq 2g$. As $|S| \leq 2g-1$ and $|\bigcup_{i=0}^{c-1} N_1(a_i)| \geq 2g$, there is at least one vertex a_i $(0 \leq i \leq c-1)$ on cycle C_1 such that there exists a vertex $v_0 \in N_1(a_i)$ not belonging to the cyclic vertex cutset S.

Note that if the component D_1 contains one subtree at the 1st level of all T-trees and the subtree is a full tree of depth $c/4-2$, then $v(D_1) \geq (3^0 + 3^1 + \cdots + 3^{c/4-2}) + c \geq (3^{c/4-1} + 1)/2$. In the following, we discuss it in three cases according to the range of c.

Case (1) $c = g$.

Note that $N_{r_1}(a_i) \cap N_{r_2}(a_j) = \emptyset$ $(0 \leq i < j \leq c-1,\ 0 < r_1, r_2 \leq c/4 - 1)$. Otherwise, it yields a cycle of length at most $c/2 + 2(c/4 - 1) = c - 2 < g$, a contradiction. So there is not any common vertex between each pair of all T-trees, i.e., $T(a_i) \cap T(a_j) = \emptyset$ $(0 \leq i < j \leq c-1)$. Furthermore, there is not any common vertex between each pair of all subtrees at the kth level of all T-trees $(0 \leq k \leq c/4-1)$. Since $N_1(a_i) \cap N_1(a_j) = \emptyset$ for any i and j, $0 \leq i < j \leq c-1$, we have $2c = 2g$ subtrees at the 1st level of all T-trees. Since $|S| \leq 2g-1$ and each vertex in S is contained in at most one subtree, at least one subtree at the 1st level of all T-trees does not contain any vertex belonging to S and is a full tree of depth $c/4 - 2$.

Case (2) $c \geq 3g - 1$.

Then there are $2c \geq 6g - 2$ subtrees at the 1st level of all T-trees. However, each vertex belonging to S is contained in at most three subtrees since G is a 4-regular graph. Then at most $3|S| \leq 6g - 3$ subtrees at the 1st level contain vertices belonging to S. Hence, at least one subtree at the 1st level of all T-trees does not contain any vertex belonging to S and the subtree is a full tree of depth $c/4 - 2$.

Case (3) $g < c \leq 3g - 2$.

Since $g \geq 19$, we have $c > g \geq 19$. Then according to Lemmas 3.1 and 3.5, we have that $v(D_1) \geq (3^{c/4-1} + 1)/2$. □

4 The proof of the correctness of Algorithm 1

In this section, the correctness of Algorithm 1 will be proved.

Let G be a 4-regular graph with the cyclic vertex cutset S, and D_1 and D_2 be two components of $G - S$, which have minimum cycles C_1 and C_2, respectively. Let $c = |V(C_1)|$. We shall prove that the number of vertices of component D_1 is at least $(3^{c/4-x_0'})/2$ (where x_0' is a positive constant). Then we have $(3^{c/4-x_0'})/2 \leq |V(D_1)| \leq v(G)$, i.e., $c \leq 4\log_3 2v + 4x_0'$. Similarly, we have $|V(C_2)| \leq 4\log_3 2v + 4x_0'$. Hence, we show the correctness of Algorithm 1 since the two inequalities $c \leq 4\log_3 2v + 4x_0'$ and $|V(C_2)| \leq 4\log_3 2v + 4x_0'$ hold.

Theorem 4.1 *For a 4-regular graph G, Algorithm* 1 *can determine the cyclic vertex connectivity $c\kappa(G)$.*

Proof Suppose that the cyclic vertex connectivity $ck(G)$ is not ∞. Then the vertices of a component D_1 in $G-S$ may be adjacent to the vertices of S, but not to those of another component D_2 in $G - S$. And the vertices of D_2 may be adjacent to vertices of S, but not to those of D_1. Let $g = 3$. Then $ck(G) \geq (2g + 2g)/4 \geq 3$ since G is 4-regular. Thus we have $v(G) \geq |V(D_1)| + ck(G) + |V(D_2)| \geq g + 3 + g \geq 9$. Obviously, the inequality $v(G) \geq 9$ also holds when $g \geq 4$. Let z denote the upper bound of lengths of all induced cycles to be considered. If $g \geq 19$, then $z = 4\log_3(2v) + 7$; if $g \leq 18$, then $z = 4\log_3(2v) + 42$. We shall discuss Algorithm 1 in two cases according to the range of v.

Case 1 $v \geq 6g$.
According to Lemma 1.1, then we have $c\kappa(G) \leq 2g$, i.e., $|S| \leq 2g$. Suppose $c\kappa(G) = 2g$. Then Algorithm 1 definitely can get the result since the value of $c\kappa(G)$ has been initialized to $2g$ in Step 2.
Now suppose that $c\kappa(G) \leq 2g - 1$, i.e., $|S| \leq 2g - 1$.

Case (1.1) $g \geq 19$.
According to Lemma 3.6, the component D_1 contains at least $(3^{c/4-1} + 1)/2$ vertices. Hence, we have that $v(G) \geq |V(D_1)| \geq (3^{c/4-1} + 1)/2$, i.e., $c \leq 4[\log_3(2v - 1) + 1] \leq 4\log_3(2v) + 7$. Algorithm 1 can get correct result since it finds all induced cycles of length less than or equal to $4\log_3(2v) + 7$.
Case (1.2) $g \leq 18$.
In Algorithm 1, we find all induced cycles of length less than or equal to $4\log_3(2v) + 42$ and get the minimum cutset between each pair of them. Since $v \geq 9$, we have that $4\log_3(2v) + 42 \geq 52.5$. So we have found all induced cycles of length less than or equal to 52. Only the upper limit of the length c of those cycles for $c \geq 53$ need be considered. Suppose that $c \geq 53$. The method of proof is the same as Case (2) in Lemma 3.6. Then there are $2c \geq 106$ subtrees at the 1st level of all T-trees. Since each vertex belonging to S is contained in at most three subtrees, at most $3|S| \leq 6g - 3 \leq 105$ subtrees at the 1st level contain vertices belonging to S. Then at least one subtree at the 1st level of all T-trees does not contain any vertex belonging to S and the subtree is a full tree of depth $c/4 - 2$. Hence, we have that $v(G) \geq |V(D_1)| \geq (3^{c/4-1} + 1)/2$, i.e., $c \leq 4[\log_3(2v - 1) + 1] \leq 4\log_3(2v) + 42$.

Similarly, if $c\kappa(G) \leq 2g-1$, then the length of the shortest cycle in component D_2 is at most $4\log_3(2v)+7$ when $g \geq 19$, and at most $4\log_3(2v)+42$ when $g \leq 18$.

Case 2 $v < 6g$.
According to Lemma 1.2, we have that $g \leq 7$ since G is a 4-regular graph. In Algorithm 1, we find all induced cycles of length less than or equal to $4\log_3(2v)+42$ and get the minimum cutset between each pair of them.

Suppose that $c\kappa(G) = \infty$. Then the value of $c\kappa(G)$ has been initialized to ∞ in Step 2 and Algorithm 1 can determine it. Suppose that $c\kappa(G) \leq 2g-1$. Then the discussion is the same as that of $c\kappa(G) \leq 2g-1$ in Case 1, and the analytical method and the result is also the same. Hence, if $g \leq 18$, then $c \leq 4\log_3(2v)+42$. Algorithm 1 can get correct result by finding all induced cycles of length less than or equal to $4\log_3(2v)+42$.

Suppose that $c\kappa(G) \geq 2g$. Then we shall prove that the induced cycles of length greater than $4\log_3(2v)+42$ cannot exist in component D_1. Since $c\kappa(G) \geq 2g$, we have that $v \geq |V(D_1)| + c\kappa(G) + |V(D_2)| \geq g+2g+g = 4g$ and $4\log_3(2v)+42 \geq 4\log_3(8g)+42$. Then we can prove that the cycles of length greater than $4\log_3(8g)+42$ cannot exist in component D_1. Suppose this type of cycle exists in D_1. Then we have

$$v \geq |V(D_1)| + c\kappa(G) + |V(D_2)| > 4\log_3(8g)+42+2g+g. \tag{4.1}$$

Since $g \leq 7$, the following inequality holds:

$$4\log_3(8g)+42+2g+g \geq 6g. \tag{4.2}$$

Then we have $v > 6g$ by (4.1) and (4.2), which contradicts the assumption that $v < 6g$.

The discussion for D_2 is the same as D_1. Hence, the upper bound z is large enough to determine the value of $c\kappa(G)$.

Therefore, if we find all induced cycles of length at most z in G and $c\kappa(G) \neq \infty$, then Algorithm 1 can find a minimum cyclic vertex cutset and determine the cyclic vertex connectivity. If $c\kappa(G) = \infty$, then Algorithm 1 also can distinguish it. □

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References

Aldred REL, Holton DA, Jackson B (1991) Uniform cyclic edge connectivity in cubic graphs. Combinatorica 11:81–96

Bondy JA, Murty USR (1976) Graph theory with applications. MacMillan Press, London

Dinitz Y (2006) Dinitz' algorithm: the original version and Even's version. In: Theoretical computer science. Springer, Berlin, pp 218–240

Dvořák Z, Kára J, Král' D, Pangrác O (2004) An algorithm for cyclic edge connectivity of cubic graphs. In: SWAT 2004, LNCS 3111, 236–247
Even S (2011) Graph algorithms. Cambridge University Press, Cambridge
Even S, Tarjan RE (1975) Network flow and testing graph connectivity. SIAM J Comput 4:507–518
Kutnar K, Marušič D (2008) On cyclic edge-connectivity of fullerenes. Discrete Appl Math 156:1661–1669
Liang J, Lou D (2018) A polynomial algorithm determining cyclic vertex connectivity of k-regular graphs with fixed k. J Comb Optim 1–11. https://doi.org/10.1007/s10878-018-0332-4
Liang J, Lou D, Zhang Z (2017) A polynomial time algorithm for cyclic vertex connectivity of cubic graphs. Int J Comput Math 94:1501–1514
Lou D, Holton DA (1993) Lower bound of cyclic edge connectivity for n-extendability of regular graphs. Discrete Math 112:139–150
Lou D, Liang K (2014) An improved algorithm for cyclic edge connectivity of regular graphs. Ars Comb 115:315–333
Lou D, Wang W (2005) An efficient algorithm for cyclic edge connectivity of regular graphs. Ars Comb 77:311–318
McCuaig WD (1992) Edge reductions in cyclically k-connected cubic graphs. J Comb Theory Ser B 56:16–44
Nedela R, Škoviera M (1995) Atoms of cyclic connectivity in cubic graphs. Math Slovaca 45:481–499
Peroche B (1983) On several sorts of connectivity. Discrete Math 46:267–277
Tait PG (1880) Remarks on the coloring of maps. Proc R Soc Edinb 10:501–503