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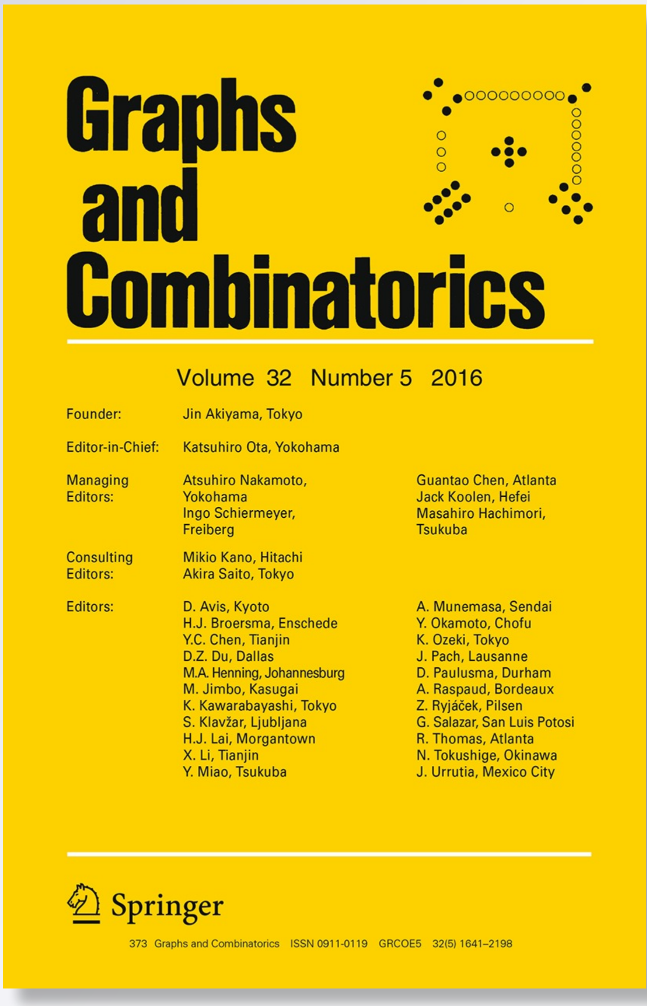

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ORIGINAL PAPER

Generalization of Matching Extensions in Graphs (IV): Closures

Hongliang Lu[1] · Qinglin Yu[2,3]

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Abstract Let G be a graph and n, k and d be non-negative integers such that $|V(G)| \geqslant n+2k+d+2$ and $|V(G)|-n-d \equiv 0 \pmod 2$. A graph is called an (n, k, d)-*graph* if deleting any n vertices from G the remaining subgraph of G contains k-matchings and each k-matching in the subgraph can be extended to a defect-d matching. We study the relationships between (n, k, d)-graphs and various closure operations, which are usually considered in the theory of hamiltonian graphs. In particular, we obtain some necessary and sufficient conditions for the existence of (n, k, d)-graphs in terms of these closures.

Keywords k-matching · Matching extension · (n, k, d)-graph · Closure

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *matching* is a set of independent edges. For a matching M, we use $V(M)$ to denote the vertices incident to the edges of M and $|M|$ to denote the number of edges in M. Let d be a non-negative integer. A matching is called a *defect-d matching* if it covers exactly $|V(G)|-d$ vertices of G. A defect-0 matching is commonly known as a *perfect matching*.

For $S \subseteq V(G)$, we write $G[S]$ for the subgraph of G induced by S and $G-S$ for $G[V(G)\backslash S]$. The number of odd components (i.e., components with odd order) of G

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is denoted by $c_o(G)$. Let $E_G(S, T)$ denote the edges of graph G with one end in S and another end in T and $e_G(S, T) = |E_G(S, T)|$, where $S, T \subseteq V(G)$. For a vertex $v \in V(G)$, $N_G(v)$ denotes the neighbourhood of v in G. For $X \subseteq V(G)$, we write $N_G(X) = \cup_{x \in X} N_G(x)$. For $x, y \in V(G)$ and $xy \notin E(G)$, let $G + xy$ denote the graph obtained from G by adding an edge xy. Given two graphs F and H, let $F \vee H$ be a graph obtained from $F \cup H$ by adding all the edges joining a vertex of F to a vertex of H.

Let M be a matching of G. If there is a matching M' such that $M \subseteq M'$, we say that M can be extended to M' or M' is an *extension* of M. A matching with the largest cardinality is called a *maximum matching* of G. We denote the *matching number*, the size of a maximum matching, by $\mu(G)$. Suppose that G is a connected graph with perfect matchings. If each k-matching (i.e., a matching with k edges) can be extended to a perfect matching in G, then G is called *k-extendable*. To avoid triviality, we require that $|V(G)| \geqslant 2k + 2$ for k-extendable graphs. This family of graphs was introduced by Plummer [7]. A graph G is called *n-factor-critical* if after deleting any n vertices the remaining subgraph of G has a perfect matching. This concept was introduced by Favaron [4] and Yu [12], independently, which is a generalization of the notions of the well-known factor-critical graphs and bicritical graphs, the cases of $n = 1$ and 2, respectively. For a given graph H, if a graph G has no induced subgraph isomorphic to H, then G is called *H-free*. The star $K_{1,3}$ is often referred as a *claw*, and so a $K_{1,3}$-free graph G is often called *claw-free*.

Let G be a connected graph and let n, k and d be non-negative integers such that $|V(G)| \geqslant n + 2k + d + 2$ and $|V(G)| - n - d \equiv 0 \pmod 2$. If deleting any n vertices from G the remaining subgraph of G contains k-matchings and moreover, each k-matching in the subgraph can be extended to a defect-d matching, then G is called an *(n, k, d)-graph*. This term was introduced by Liu and Yu [6] as a general framework to unify the concepts of defect-d matchings, n-factor-criticality and k-extendability. In particular, $(n, 0, 0)$-graphs are exactly n-factor-critical graphs and $(0, k, 0)$-graphs are the same as k-extendable graphs.

Bondy and Chvátal [3] defined the r-closure $c_r(G)$ as the graph obtained from G by recursively joining pairs of nonadjacent vertices the degree sum of which is at least r until no such pair remains. The r-closure $c_r(G)$ has a strong connection with hamiltonian cycles, as shown in the next theorem.

Theorem 1.1 (Bondy and Chvátal, [3]) *Let G be a graph of order $n \geqslant 3$. Let x and y be a pair of distinct nonadjacent vertices of G with $d_G(x) + d_G(y) \geqslant n$. Then G is hamiltonian if and only if $G + xy$ is hamiltonian.*

Following this notion, a number of other types of closures have been introduced. A vertex x of a graph G is said to be *locally r-connected* if $N_G(x)$ induces an r-connected graph in G. A locally r-connected vertex x is said to be *r-eligible* if $N_G(x)$ induces a non-complete graph. For a locally r-connected vertex x of a graph G, we consider the operation of joining every pair of nonadjacent vertices in $N_G(x)$ by an edge so that $N_G(x)$ induces a complete subgraph in the resulting graph. This operation is called the *local completion* of G at a locally r-connected vertex x. We consider a sequence of local completions $G = G_0, G_1, \ldots, G_m = H$, where G_{i+1} is obtained from G_i by a local completion at a locally r-connected vertex for each i, $0 \leqslant i \leqslant m - 1$. If H does not have an r-eligible locally r-connected vertex, then H is called an *r-closure* of G and denoted by $cl_r(G)$. Ryjáček [10] introduced closure $cl_1(G)$ and proved that a claw-free graph G is hamiltonian if and only if $cl_1(G)$ is hamiltonian. Bollobás et al. generalized it to $cl_r(G)$ in [2] and proved that $cl_r(G)$ is uniquely determined for each r and G is hamiltonian-connected if and only if $cl_3(G)$ is hamiltonian-connected.

Plummer and Saito [9] gave necessary and sufficient conditions for a graph to be n-factor-critical in terms of these closures. They also investigated the relationships between the various closures and matching extension.

In this paper, we further study the relationships between various closures and (n, k, d)-graphs. In the next section, we give necessary and sufficient conditions for (n, k, d)-graphs in terms of Bondy-Chvátal-type closure. In Sect. 3, we study a closure based on neighborhood unions. And in Sect. 2, we study Ryjáček's closure and present two sufficient conditions for claw-free (n, k, d)-graphs.

The proofs of the main theorems require the following results.

Theorem 1.2 (Berge [1]) *Let G be a graph and d an integer such that $0 \leqslant d \leqslant |V(G)|$ and $|V(G)| \equiv d \pmod 2$. Then G has no defect-d matchings if and only if there exists a vertex subset $S \subseteq V(G)$ such that*

$$c_o(G - S) \geqslant |S| + d + 2.$$

In [6], Liu and Yu obtained the following necessary and sufficient conditions for (n, k, d)-graphs.

Theorem 1.3 (Liu and Yu [6]) *A graph G is an (n, k, d)-graph if and only if the following conditions hold:*

(a) for any $S \subseteq V(G)$ such that $|S| \geqslant n$,

$$c_o(G - S) \leqslant |S| - n + d,$$

(b) for any $S \subseteq V(G)$ such that $|S| \geqslant n + 2k$ and $G[S]$ contains a k-matching,

$$c_o(G - S) \leqslant |S| - n - 2k + d.$$

Sumner [11] considered perfect matchings in claw-free graphs.

Theorem 1.4 (Sumner [11]) *A connected claw-free graph of even order has a perfect matching.*

From the above result, it is easy to see the next one.

Corollary 1.5 *A connected claw-free graph of odd order has a defect-1 matching.*

2 Bondy-Chvátal-Type Closure

Theorem 2.1 *Let G be a graph of order p, and x, y a pair of distinct nonadjacent vertices of G with $d_G(x) + d_G(y) \geqslant p + n - d - 1$. Then G is an $(n, 0, d)$-graph if and only if $G + xy$ is an $(n, 0, d)$-graph.*

Proof By the definition of (n, k, d)-graphs, the necessity is obvious. So we prove the sufficiency. Suppose that $G + xy$ is an $(n, 0, d)$-graph but G is not an $(n, 0, d)$-graph. By Theorem 1.3 and parity, there exists a vertex subset S of G with $|S| \geqslant n$ such that

$$c_o(G - S) \geqslant |S| - n + d + 2.$$

Since $G + xy$ is an $(n, 0, d)$-graph, x and y must belong to different odd components of $G - S$, say C_i and C_j. However, every odd component of $G - S$ contains at least one vertex, so

$$\begin{aligned} d_G(x) + d_G(y) &\leqslant (|C_i| - 1) + (|C_j| - 1) + 2|S| \\ &\leqslant p - |S| - (|S| - n + d) - 2 + 2|S| \\ &\leqslant p + n - d - 2, \end{aligned}$$

a contradiction. □

Theorem 2.2 *Let n, k, d be nonnegative integers. Let G be a graph of order p, and x, y a pair of distinct nonadjacent vertices of G with $d_G(x)+d_G(y) \geqslant p+n+2k-d-1$. If $G + xy$ is an (n, k, d)-graph, then G is an (n, k, d)-graph.*

Proof Suppose that G is not an (n, k, d)-graph. By the definition of (n, k, d)-graphs, there exists a vertex subset S of G with $|S| = n$ and a k-matching M of $G - S$ such that $G - S - V(M)$ contains no matchings of deficiency d. So by Theorem 1.2, there exists a subset $W \subseteq V(G - S - V(M))$ such that $c_o(G - S - V(M) - W) \geqslant w + d + 2$, where $|W| = w$. Let $q = c_o(G - S - V(M) - W)$ and $C_1, \ldots, C_q$ denote those odd components of $G - S - V(M) - W$. Without loss of generality, we assume that $|C_1| \leqslant \cdots \leqslant |C_q|$. We choose two vertices $u, v \in V(G) - S - V(M) - W$ such that u, v belong to different odd components, say C_i and C_j. Since $p \geqslant |C_{q-1}| + |C_q| + (q - 2) + n + 2k + w$ and $q \geqslant w + d + 2$, we have

$$\begin{aligned} d_G(u) + d_G(v) &\leqslant (|C_i| - 1) + (|C_j| - 1) + 2n + 4k + 2w \\ &\leqslant (|C_{q-1}| - 1) + (|C_q| - 1) + 2n + 4k + 2w \\ &\leqslant p - (w + d) - 2 + n + 2k + w \\ &\leqslant p + n + 2k - d - 2. \end{aligned}$$

Hence x, y cannot belong to two different odd components of $G - S - V(M) - W$. So $c_o((G + xy) - S - V(M) - W) \geqslant w + d + 2$ and by Theorem 1.3, $G + xy$ is not an (n, k, d)-graph, a contradiction. This completes the proof. □

Remark Theorem 2.2 is best possible in the following sense. For nonnegative integers n, k, l and d. Let $G = K_{n+2k+l} \vee (l + d + 2)K_1$. By Theorem 1.3, G is not an (n, k, d)-graph, but $G + xy$ is an (n, k, d)-graph for any $xy \notin E(G)$. Furthermore, $d_G(x) + d_G(y) = p + n + 2k - d - 2$ for all $xy \notin E(G)$.

The converse of Theorem 2.2 does not hold if $k > 0$. (If $k = 0$, then the converse holds by Theorem 2.1.) Let $G = (n+k+1)K_1 \vee (K_{2m+k} \cup (d+1)K_{2m+2k+1})$, where m

is a sufficiently large integer. Then $p = |V(G)| = 2(d+2)m+2(d+2)k+n+d+2 \equiv n + d \pmod 2$. Let M be a matching of G with $|M| = k$ and S be a subset of $V(G - V(M))$ with $|S| = n$. Then $G - S - V(M) = bK_1 \vee \big(K_a \cup (\cup_{i=1}^{d+1} K_{c_i})\big)$ for suitable integers a, b and c_i $(1 \leqslant i \leqslant d+1)$, where $b > 1$. Therefore, $G - S - V(M)$ has a matching of deficiency d, and hence G is an (n, k, d)-graph. Let u and v be two distinct vertices in $(n+k+1)K_1$. Then u and v are not adjacent, and $d_G(u)+d_G(v) = 2(2m+k+(d+1)(2m+2k+1)) = 2p-2(n+k+1) > p+n+2k-d-1$. Let S' be a subset of $V((n+k+1)K_1 - u - v)$ with $|S'| = n$ and M' be a k-matching of $G+uv$ which consists of uv and $k-1$ independent edges joining $(n+k+1)K_1 - S' - u - v$ and K_{2m+k}. Since $(G + uv) - S' - V(M')$ contains $(d + 2)$ odd components, by Theorem 1.3, $G + uv$ is not an (n, k, d)-graph.

If we enhance the condition in Theorem 2.2 by requesting that the same degree condition holds for *all* pairs of x, y with $uv \notin E(G)$, then the converse of Theorem 2.2 is true. In fact, we can prove the following stronger result. Note that an (n, k, d)-graph G implies that $G + xy$ is also an (n, k, d)-graph for any pair of vertices x, y.

Theorem 2.3 *Let n, k, d be nonnegative integers and G be a graph of order p. If $d_G(x) + d_G(y) \geqslant p + n + 2k - d - 1$ for any pair of non-adjacent vertices x and y, then G is an (n, k, d)-graph.*

Proof Suppose that the conclusion does not hold. Then there exists a vertex subset $R \subseteq V(G)$ of order n and a k-matching M of $G - R$ such that $G - R - V(M)$ has no defect-d matchings. By Theorem 1.2, there exists a vertex subset S of $G - R - V(M)$ such that

$$q = c_o(G - R - V(M) - S) \geqslant |S| + d + 2. \tag{1}$$

Let $C_1, \ldots, C_q$ be all odd components of $G - R - V(M) - S$ such that $|C_1| \leq |C_2| \leq \cdots \leq |C_q|$ and let $u \in C_1, v \in C_2$. Since $p \geqslant |C_1| + |C_2| + n + 2k + |S| + (|S| + d)$, we have

$$\begin{aligned} d_G(u) + d_G(v) &\leqslant (|C_1| - 1) + (|C_2| - 1) + 2(n + 2k + |S|) \\ &\leqslant p + n + 2k - d - 2, \end{aligned}$$

a contradiction. This completes the proof. □

By Theorem 2.1, we have the following corollary.

Corollary 2.4 (Plummer and Saito [9]) *Let n be a nonnegative integer, G be a graph of order p, and x, y be a pair of distinct nonadjacent vertices of G with $d_G(x)+d_G(y) \geqslant p + n - 1$. Then $G + xy$ is n-factor-critical if and only if G is n-factor-critical.*

Theorem 2.2 implies the following result.

Corollary 2.5 (Plummer and Saito [9]) *Let k be a nonnegative integer, G be a graph of order p, and x, y be a pair of distinct nonadjacent vertices of G with $d_G(x)+d_G(y) \geqslant p + 2k - 1$. If $G + xy$ is k-extendable, then G is k-extendable.*

Theorem 2.3 is an extension of the following two theorems.

Theorem 2.6 (Plummer [8]) *Let k be a nonnegative integer and G a graph of order p. If $d_G(x) + d_G(y) \geqslant p + 2k - 1$ for any two nonadjacent vertices x, y, then G is k-extendable.*

Theorem 2.7 (Favaron [4]) *Let n be a nonnegative integer and G a graph of order p. If $d_G(x) + d_G(y) \geqslant p + n - 1$ for any two nonadjacent vertices of G, then G is n-factor-critical.*

3 Neighborhood Unions

Theorem 3.1 *Let n, k, d be three nonnegative integers such that $n \geqslant d$. Let G be an m-connected graph of order p, and x, y be a pair of distinct nonadjacent vertices of G with $|N_G(x) \cup N_G(y)| \geqslant p + n + 2k - m - d - 1$. Then $G + xy$ is an (n, k, d)-graph if and only if G is an (n, k, d)-graph.*

Proof We firstly prove the necessity. Assume $G + xy$ is an (n, k, d)-graph, but G is not an (n, k, d)-graph. By the definition of (n, k, d)-graphs, there exists a subset $R \subseteq V(G)$ of order n and a k-matching M in $G - R$ such that $G - R - V(M)$ has no defect-d matchings. By Theorem 1.2, there exists a subset $S \subseteq V(G - R - V(M))$ such that

$$q = c_o(G - R - V(M) - S) \geqslant |S| + d + 2.$$

Denote the odd components by $C_1, \ldots, C_q$ such that $|C_1| \leqslant \cdots \leqslant |C_q|$. By the hypothesis that $G + xy$ is an (n, k, d)-graph, x and y belong to different odd components C_i and C_j of $G - R - V(M) - S$. Since G is m-connected, $|R \cup V(M) \cup S| \geqslant m$. Moreover,

$$|N_G(x) \cup N_G(y)| \leqslant |C_i| + |C_j| - 2 + |R \cup V(M) \cup S|.$$

On the other hand, since each of the other $q - 2$ odd components of $G - (R \cup V(M) \cup S)$ contains at least one vertex, we have

$$\begin{aligned} |N_G(x) \cup N_G(y)| &\geqslant p + n + 2k - m - d - 1 \\ &\geqslant |C_i| + |C_j| + |R \cup V(M) \cup S| + (q - 2) + n + 2k - m - d - 1 \\ &\geqslant |C_i| + |C_j| + |R \cup V(M) \cup S| + (|S| + d) + n + 2k - m - d - 1 \\ &= |C_i| + |C_j| + |R \cup V(M) \cup S| + |S| + n + 2k - m - 1. \end{aligned}$$

So we have $|S| + n + 2k \leqslant m - 1$, a contradiction to $|R \cup V(M) \cup S| \geq m$.

Now we prove the sufficiency. Assume that G is an (n, k, d)-graph but $G + xy$ is not an (n, k, d)-graph. By Theorem 1.2, there exists a subset $R \subseteq V(G)$ of order at least n and a k-matching M in $(G + xy) - R$ such that

$$q = c_o((G + xy) - R - V(M)) \geqslant |R| - n + d + 2. \tag{2}$$

Let $H = G[R \cup V(M)]$. By the hypothesis that $G + xy$ is not an (n, k, d)-graph, it implies $xy \in M$; otherwise, by (2) and Theorem 1.3, then G is not an (n, k, d)-graph. Furthermore, we have $\mu(H) = k - 1$ and $\mu(H + xy) = k$. By Theorem 1.2, there exists a subset $S \subseteq V(H)$ such that

$$c_o(H - S) = |H| - 2\mu(H) + |S|.$$

Since $\mu(H + xy) > \mu(H) = k - 1$, x and y belong to different odd components of $H - S$, say C'_i and C'_j. Let M' be a maximum matching of $H + xy$. We have $xy \in M'$. Moreover, M' covers every vertex of $(V(C'_i) \cup V(C'_j) \cup S) - x - y$. Hence

$$|N_H(x) \cup N_H(y)| \leqslant |C'_i| + |C'_j| - 2 + |S| \leqslant 2(k - 1).$$

Therefore,

$$|N_G(x) \cup N_G(y)| \leqslant 2(k - 1) + |V(G - R - V(M))|.$$

Recall that $|N_G(x) \cup N_G(y)| \geqslant p + n + 2k - m - d - 1$. So we have

$$|R \cup V(M)| \leqslant m - n + d - 1 \leqslant m - 1,$$

a contradiction to the connectivity of G. □

Remark Theorem 3.1 is best possible in the following sense. Let k, n, m and d be nonnegative integers such that $m \geqslant n + 2k$. Let $G = K_m \vee (m - n - 2k + d + 2)K_1$. By Theorem 1.3, G is not an (n, k, d)-graph, but $G + xy$ is an (n, k, d)-graph for any $xy \notin E(G)$. Furthermore, $|N_G(x) \cup N_G(y)| = p + n + 2k - m - d - 2$ for any nonadjacent vertices x, y.

4 Ryjáček's Closure

For claw-free graphs, we firstly prove the following result.

Theorem 4.1 *Let $n + 2k \leqslant d$ and $p \equiv n + d \pmod 2$. If G is a connected claw-free graph of order p, then G is an $(n + 2k, 0, d)$-graph.*

Proof We only consider odd p and the proof is similar for even p. Since $p \equiv n + d \pmod 2$, we have $n \not\equiv d \pmod 2$ and so $n + 2k < d$. Since G is a connected claw-free graph, by Corollary 1.5, G contains a matching of deficiency one. So for any subset $S \subseteq V(G)$, $G - S$ contains a matching of deficiency at most $|S| + 1$. As $n + 2k < d$, G is an $(n + 2k, 0, d)$-graph. □

By Theorems 1.3 and 4.1, the following result is immediate.

Corollary 4.2 *Let $n + 2k \leqslant d$ and $p \equiv n + d \pmod 2$. If G is a connected claw-free graph of order p, then G is an (n, k, d)-graph.*

For Ryjáček closure, Plummer and Saito proved the following result.

Theorem 4.3 (Plummer and Saito, [9]) *Let G be a claw-free graph and x a locally n-connected vertex. Let G' be the graph obtained from G by local completion at x in G. Then G is n-factor-critical if and only if G' is n-factor-critical.*

As a generalization of the above theorem, we prove the following result for (n, k, d)-graphs.

Theorem 4.4 *Let $n+2k > d$ and G be a 2-connected claw-free graph. If $cl_{n+2k-d}(G)$ is an (n, k, d)-graph, then G is an (n, k, d)-graph.*

In fact, we prove the following theorem, which clearly implies Theorem 4.4.

Theorem 4.5 *Let $n + 2k > d$. Let G be a 2-connected claw-free graph and x a locally $(n + 2k - d)$-connected vertex. Let G' be the graph obtained from G by local completion at x in G. If G' is an (n, k, d)-graph, then G is also an (n, k, d)-graph.*

Proof Assume G' is an (n, k, d)-graph but G is not an (n, k, d)-graph. Then there exists a subset $S \subseteq V(G)$ with $|S| = n$ and a k-matching M of $G - S$ such that $G - S - V(M)$ contains no matchings of deficiency d. Since G is claw-free, clearly $G - S - V(M)$ is also claw-free. So by Theorem 1.4 and Corollary 1.5, $G - S - V(M)$ contains at least $d+2$ odd components. Let $C_1, \ldots, C_q$ denote all the odd components of $G - S - V(M)$, where $q \geqslant d + 2$.

If $x \notin S \cup V(M)$, then x belongs to a component, say C_1, of $G - S - V(M)$. Then $N_G(x) \subseteq V(C_1) \cup S \cup V(M)$ and hence $G' - S - V(M)$ has the same number of odd components as $G - S - V(M)$, a contradiction to that G' is an (n, k, d)-graph.

So we assume $x \in S \cup V(M)$. Furthermore, there exists two odd components of $G - S - V(M)$, say C_1 and C_2, such that $N_G(x) \cap V(C_1) \neq \emptyset$ and $N_G(x) \cap V(C_2) \neq \emptyset$. Let $x_1 \in N_G(x) \cap V(C_1)$ and $x_2 \in N_G(x) \cap V(C_2)$. Then x_1 and x_2 are separated in the graph induced by $N_G(x) - S - V(M)$. We shrink each C_i into a vertex u_i and denote $T = \{u_i \mid 1 \leqslant i \leqslant q\}$. Now we construct a (simple) bipartite graph H with vertex set $V(H) = T \cup S \cup V(M)$ and edge set $E(H) = \{vu_i \mid v \in (S \cup V(M)) \cap N_G(C_i) \text{ and } u_i \in T\}$. Since G is a 2-connected claw-free graph, $e_H(v, T) \leqslant 2$ for all $v \in S \cup V(M)$, $e_H(u, S \cup V(M)) \geqslant 2$ for all $u \in T$ and $e_H(x, T - u_1 - u_2) = 0$. So we have

$$e_H(T, S \cup V(M)) \leqslant 2(n + 2k). \tag{3}$$

Note that $G[N_G(x)]$ is $(n + 2k - d)$-connected, by Menger's Theorem, there exists at least $n + 2k - d$ disjoint paths of $G[N_G(x)]$ from x_1 to x_2. So we have $e_H(u_i, S \cup V(M) - x) \geqslant n + 2k - d$ for $i = 1, 2$. Now we obtain

$$\begin{aligned} e_H(T, S \cup V(M)) &\geqslant \sum_{i=1}^{2} e_H(u_i, S \cup V(M)) + e_H(T - \{u_1, u_2\}, (S \cup V(M)) - x) \\ &\geqslant 2(n + 2k - d + 1) + 2d \\ &= 2(n + 2k) + 2, \end{aligned}$$

a contradiction to (3). This completes the proof. $\square$

By Theorems 1.3 and 4.5, it is easy to see the following.

Corollary 4.6 *Let G be a 2-connected claw-free graph and x a locally n-connected vertex. Let G' be the graph obtained from G by local completion at x in G. Then G' is an $(n, 0, d)$-graph if and only if G is an $(n, 0, d)$-graph.*

By similar arguments, when $d = 0$, we can remove the 2-connectivity condition of G in Theorem 4.5 to obtain the following result.

Theorem 4.7 *Let G be a claw-free graph and x a locally $(n + 2k)$-connected vertex. Let G' be the graph obtained from G by local completion at x in G. If G' is an $(n, k, 0)$-graph, then G is also an $(n, k, 0)$-graph.*

Remark The converse of Theorem 4.7 does not hold for $n + 2k > 0$. We only give an example for even n since it can be discussed similarly for odd n. Let m be a sufficient large odd integer. Let H_1 and H_2 be two copies of $K_{(n+1)m}$. Let $F = K_{2k-1}$ and $V(F) = \{u_1, \ldots, u_{2k-1}\}$. We write $V(H_q) = \{x^q_{ij} \mid 1 \leqslant i \leqslant n + 1 \text{ and } 1 \leqslant j \leqslant m\}$ for $q = 1, 2$. Let $S = \{v_i \mid 1 \leqslant i \leqslant n + 1\}$. Now we construct a graph G with vertex set $V(G) = S \cup V(F) \cup V(H_1) \cup V(H_2)$ and edge set $E(G) = E(H_1) \cup E(H_2) \cup E(K_{2k-1}) \cup E_1 \cup E_2$, where $E_1 = \{v_i x^q_{ij} \mid 1 \leqslant q \leqslant 2, 1 \leqslant i \leqslant n+1 \text{ and } 1 \leqslant j \leqslant m\}\}$ and $E_2 = \{ux \mid u \in V(F) \text{ and } x \in V(H_1) \cup V(H_2)\}$. Then G is claw-free and G is also an $(n, k, 0)$-graph. Note that $G[N_G(x^1_{11})]$ is $(n + 2k)$-connected. However, if we apply local completion at x^1_{11}, the resulting graph G' is not an $(n, k, 0)$-graph, since $G'[S \cup V(F)]$ contains a k-matching but $G' - S - V(F)$ contains two odd components.

We believe that the 2-connectivity condition in Theorem 4.5 is unnecessary. However, we cannot find a way to avoid it in the proof. So we leave it here as a problem for the interested readers to consider.

Problem 4.8 *Does Theorem 4.5 still hold without 2-connectivity condition for G?*

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