Threshold For Matching-Covered Graphs*

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Abstract

A graph G is *matching-covered* (or 1-*extendable*) if it is connected and every edge of G is contained in a perfect matching. We call a graph G *bicritical* if $G - x - y$ has a perfect matching for every pair of distinct vertices x and y in G. A bicritical graph is called *brick* if it is 3-connected. In this paper, we determine thresholds for bicritical graphs and matching-covered bipartite graphs. For non-bipartite matching-covered graphs, we find a probability sequence which acts the same way like a threshold. Furthermore, we show that asymptotically almost surely all 3-connected graphs are bricks.

Key words: matching-covered, brick, bicriticality, threshold, negatively related.

1 Introduction

Let $G = (V, E)$ be a graph. A *matching* M of G is a subset of $E(G)$ such that any two edges of M have no vertices in common. A *perfect matching* (or *pm*, in short) is a matching incident with every vertex of G. We call a graph G *matching-covered* (or 1-*extendable*) if it is connected and every edge of G is contained in a perfect matching. Matching-covered graphs are well-studied [13] and it plays an important role in the study of matching theory.

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A graph G is *k-factor-critical* if $G - S$ has a perfect matching for any k-subset S of $V(G)$. For the cases of $k = 1, 2$, they are referred as *factor-critical* and *bicritical* by Gallai and Lovász (see [13]), respectively. A *brick* is a 3-connected bicritical graph. The factor-critical graphs are used as essential "building blocks" for the so-called Gallai-Edmonds matching structure of general graphs and bricks are studied by Lovász to develop a brick-decomposition as a powerful tool to determine the dimension of matching lattice.

The notion of a random graph was originated by Erdős and Rényi in 1947. They investigated the thresholds for k-connectivity and perfect matchings. Their results revealed that as soon as the last isolated vertex disappears, the random graph becomes connected. Moreover, provided $|V(G)|$ is even, from that very moment the random graph also contains a perfect matching.

There are two basic models for random graphs, the binomial model and the uniform model. Let Ω be the set of all graphs on vertex set $[n] = \{1, 2, \ldots, n\}$ and $e_G = |E(G)|$ stands for the number of edges of G. Given a real number p, $0 \leq p \leq 1$, the *binomial random graph*, denoted by $\mathbb{G}(n, p)$, is the set Ω of graphs G with probability $\mathbb{P}(G) = p^{e_G}(1-p)^{\binom{n}{2} - e_G}$. Given an integer M, $0 \leq M \leq \binom{n}{2}$, the *uniform random graph*, denoted by $\mathbb{G}(n, M)$, is defined by the family of all graphs G on the vertex set $[n]$ with exactly M edges, and $\mathbb{P}(G) = (\binom{\binom{n}{2}}{M})^{-1}$. Let Γ be a finite set, $|\Gamma| = N$, $0 \leq p \leq 1$, we call Γ_p *the random subset* of Γ, if it has probability distribution on $\Omega = 2^{\Gamma}$ given by $\mathbb{P}(F) = p^{|F|}(1-p)^{N-|F|}$, for any $F \subseteq \Gamma$. In this paper, we use the notation $a_n \backsim b_n$ if $a_n / b_n \rightarrow 1$.

A family of subsets $\mathcal{Q} \subseteq 2^{\binom{n}{2}}$ is called *increasing* if $A \subseteq B$ and $A \in \mathcal{Q}$ imply that $B \in \mathcal{Q}$. A family of subsets is *decreasing* if the family of the complements in Γ is increasing. A family which either increasing or decreasing is called *monotone*. We identify properties of subsets of Γ with the corresponding families of all subsets having the property.

For an increasing famliy $\mathcal{Q}$, a sequence $\hat{p} = \hat{p}(n)$ is called a *threshold* if

$$\mathbb{P}(\Gamma_p \in \mathcal{Q}) \longrightarrow \begin{cases} 0 & \text{if } p \ll \hat{p}, \\ 1 & \text{if } p \gg \hat{p}. \end{cases}$$

Thresholds for decreasing properties are defined as the thresholds of their complements. Note that both matching-covering in bipartite graphs and bicriticality are increasing properties, so we attempt to determine their thresholds. However, matching-covering in non-bipartite graphs is not a monotone property, so the threshold is not defined. We note that matching-covering in non-bipartite graphs possesses the similar phenomenon as the threshold, that is, there exists a sequence c_n satisfies the given conditions such that, the limiting probability that a random graph is matching-covered jumps from 0 to 1 very rapidly, with a rather tiny increase in the number of edges. Moreover, we determine an exact probability distribution with the desired c_n.

The expected value and the variance of a random variable X are denoted by $\mathbb{E}X$ and $VarX$, respectively. We denote the covariance of two random variables X and Y by $Cov(X, Y)$. In the proofs of the main theorems, we deploy the well-known *Chebyshev's* *inequality*: if $VarX$ exists, then $\mathbb{P}(|X - \mathbb{E}X| \geq t) \leq \frac{VarX}{t^2}$ (for $t > 0$), and *Markov's inequality*: if $X \geq 0$ almost surely, then $\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}X}{t}$ (for $t > 0$). We write *asymptotically almost surely* as *a.a.s.* in short. The *total variation distance* between the distributions of two random variables X and Y is, in general, defined by

$$d_{TV}(X, Y) = \sup_{A} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|,$$

takeing the super bound over all Borel sets A.

The following well-known inequalities are used in the next section.

$$\binom{n}{r} \leq \frac{n^r}{r!} \quad (r = 1, 2 \ldots;\ n = r, r+1, \ldots) \tag{1}$$

$$n! \geq \left(\frac{n}{e}\right)^n \text{ for } n \geq 1. \tag{2}$$

$$\text{If } \lambda > 1,\ 0 < \delta < 1/\lambda e, \text{ then } \sum_{ne\delta\lambda \leq r \leq n} \binom{n}{r} \delta^r = O(1/\lambda^{ne\delta\lambda}) \tag{3}$$

2 Preliminary Results

We start with several known results which will be used in our proofs of main theorems.

The proofs are heavily relied on the characterizations of matchings in bipartite graphs (i.e., Hall's Theorem) and matching-covered bipartite graphs (i.e., Theorem 3).

Theorem 1. (Hall's Theorem, [9]) *A bipartite graph $G = (U, W)$ has a perfect matching if and only if $|U| = |W|$ and $|N(S)| \geq |S|$, for any set $S \subseteq U$.*

Lemma 1. (Lovász, [12]) *A graph G is bicritical if and only if $q(G - S) \leq |S| - 2$ for any $S \subseteq V(G)$, where $q(G - S)$ denotes the number of odd components of $G - S$.*

Lemma 2. (Plummer, [15]) *If G is matching-covered, then it is 2-connected.*

Lemma 3. (Lovász, [12]) *A bipartite graph $G = (U, W)$ is matching-covered if and only if $|U| = |W|$ and $|N(X)| \geq |X| + 1$ for any $\emptyset \neq X \subset U$.*

To determine the sequence c_n for matching-covered graphs and bricks, we follow the similar approaches developed by Erdős and Rényi [6, 7, 8] for thresholds of perfect matchings in bipartite graphs and general graphs, which are presented below.

Lemma 4. (Erdős and Rényi, [6]) *Let $c_n = np - \log n$ and let T_1 be the number of isolated vertices in $\mathbb{G}(n, p)$. Then*

$$\mathbb{P}(T_1 > 0) \to \begin{cases} 0 & \textit{if } c_n \to \infty, \\ 1 & \textit{if } c_n \to -\infty. \end{cases}$$

Lemma 5. (Erdős and Rényi, [7]) *Let* $c_n = np - \log n$ *and* $\mathbb{G}(n,n,p)$ *be a random bipartite graph with bipartition* (V_1, V_2)*, where* $|V_1| = |V_2| = n$*. Then*

$$\mathbb{P}(\mathbb{G}(n,n,p) \text{ has a pm}) \longrightarrow \begin{cases} 0 & \text{if } c_n \to -\infty, \\ e^{-2e^{-c}} & \text{if } c_n \to c, \\ 1 & \text{if } c_n \to \infty. \end{cases}$$

Lemma 6. (Erdős and Rényi, [8]) *Let* $c_n = np - \log n$*. Then*

$$\mathbb{P}(\mathbb{G}(n,p) \text{ has a pm}) \longrightarrow \begin{cases} 0 & \text{if } c_n \to -\infty, \\ e^{-e^{-c}} & \text{if } c_n \to c, \\ 1 & \text{if } c_n \to \infty. \end{cases}$$

Let W be a set and $A \subseteq W$ be a random event. Suppose that $X = \sum_{\alpha \in A} I_\alpha$, where I_α is a random indicator variable and suppose that, for each $\alpha \in A$, there is a family of random indicator variables J_{β_α}, $\beta \in A \setminus \{\alpha\}$ such that the distribution $\mathcal{L}$ satisfies the condition

$$\mathcal{L}(\{J_{\beta_\alpha}\}_\beta) = \mathcal{L}(\{I_\beta\}_\beta \mid I_\alpha = 1) \tag{4}$$

that is, the joint distribution of $\{J_{\beta_\alpha}\}_\beta$ equals to the conditional distribution of $\{I_\beta\}_\beta$ given $I_\alpha = 1$. Then we say that the random indicator variables $(I_\alpha)_{\alpha \in A}$ are *positively related* if , for each $\alpha \in A$, there exist random variables J_{β_α} with the distribution (4), such that $J_{\beta_\alpha} \geq I_\beta$ for every $\beta \neq \alpha$.

Following lemmas are used in the proofs of the main theorems.

Lemma 7. (Barbour, Holst and Janson, [1]) *Suppose that* $X = \sum_{\alpha \in A} I_\alpha$*, where* I_α *are positively related random indicator variables. Then, with* $\pi_\alpha = \mathbb{E} I_\alpha$ *and* $\lambda = \mathbb{E}X = \sum_{\alpha \in A} \pi_\alpha$*,*

$$d_{TV}(X, Po(\lambda)) \leq \min\{\lambda^{-1}, 1\} \left(VarX - \mathbb{E}X + 2\sum_{\alpha \in A} \pi_\alpha^2 \right) \leq \frac{VarX}{\mathbb{E}X} - 1 + 2 \max_{\alpha \in A} \pi_\alpha.$$

Lemma 8. (Barbour, Holst and Janson, [1]) *Suppose that the indicator variables* $\{I_\alpha\}_{\alpha \in A}$ *are all increasing functions of some underlying independent random variables* $\{Y_j\}$*. Then the variables* $\{I_\alpha\}_{\alpha \in A}$ *are positively related.*

Lemma 9. (Ivchenko, [10]) *Let* $k \geq 2$*,* $\omega(n) \to \infty$*,* $\omega(n) \leq \log\log\log n$ *and* $p = (\log n + (k-1)\log\log n - \omega(n))/n$*. Then there are almost no* $\mathbb{G}(n,p)$ *containing a non-trivial* $(k-1)$*-cutset.*

Lemma 10. (Bollobás and Thomason, [5]) *Let* $k \in N$*,* x *be a fixed real number and* $M(n) = \frac{n}{2}(\log n + k\log\log n + x + o(1)) \in N$*. Then* $\mathbb{P}(\kappa(\mathbb{G}(n,M)) = k) \longrightarrow 1 - e^{-\frac{e^{-x}}{k!}}$ *and* $\mathbb{P}(\kappa(\mathbb{G}(n,M)) = k+1) \longrightarrow e^{-\frac{e^{-x}}{k!}}$*, where* $\kappa(G)$ *is the vertex connectivity of* G*.*

3 Random Bipartite Matching-Covered Graphs

In the proofs of the main theorems, we apply the First and the Second Moment Methods. For any non-negative integer valued random variable X, Markov's inequality $\mathbb{P}(X > 0) \leq \mathbb{E}X$ holds. The First Moment Method relies on showing that $\mathbb{E}X = o(1)$, and thus concludes that $X = 0$ *a.a.s.* The Second Moment Method is based on Chebyshev's inequality, which implies that for any random variable X with $\mathbb{E}X > 0$, $\mathbb{P}(X = 0) \leq \frac{VarX}{(\mathbb{E}X)^2}$. Hence, by showing that the right-hand side of the inequality converges to 0, one concludes that $X > 0$ *a.a.s.*

Theorem 2. *Assume that* $\log n + \frac{1}{2}\log\log n \leq np \leq 2\ \log n$, *let* $c_n = np - \log n - \log\log n$. *Then*

$$\mathbb{P}(\mathbb{G}(n,n,p) \textit{ is matching-covered}) \longrightarrow \begin{cases} 0 & \textit{if } c_n \to -\infty, \\ e^{-2e^{-c}} & \textit{if } c_n \to c, \\ 1 & \textit{if } c_n \to \infty. \end{cases}$$

Proof. Suppose that the random bipartite graph $\mathbb{G}(n,n,p)$ with bipartition (V_1, V_2) is not matching-covered. Then, by Lemma 3, there exists a vertex set S, where $\emptyset \neq S \subset V_i$ for some $i = 1, 2$, such that $|N(S)| \leq |S|$. On the other hand, $\log n + \frac{1}{2}\log\log n \leq np \leq 2\ \log n$, together with Lemma 5, we see that $\mathbb{G}(n,n,p)$ *a.a.s.* has a perfect matching. By Hall's Theorem, $|N(S)| \geq |S|$. Thus, $|N(S)| = |S|$ *a.a.s.* Let S be a minimal such set, and $s = |S|$.

If $s = 1$, then S is a leaf.

If $s \geq 2$, then, by minimality, S satisfies

(i) $|N(S)| = |S|$,

(ii) $|S| \leq \lfloor \frac{n}{2} \rfloor$,

(iii) every vertex in S is adjacent to at least two vertices of $N(S)$.

Let A denote the event that there is a minimal set S satisfying (i)-(iii), we obtain

$$\begin{aligned} \mathbb{P}(A) &\leq \sum_{s\geq 2}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{s}\binom{n}{s}\binom{s}{2}^s p^{2s}(1-p)^{s(n-s)} \\ &\leq \binom{n}{2}\binom{n}{2} p^4(1-p)^{2(n-2)} + \sum_{s\geq 3}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{s}\binom{n}{s}\binom{s}{2}^s p^{2s}(1-p)^{s(n-s)} \\ &= o(1). \end{aligned}$$

In conclusion, the threshold of matching-covered bipartite graph coincides with that of the disappearance of leaves. For any $\alpha \in V(\mathbb{G}(n,n,p))$, we define an indicator variable I_α:

$$I_\alpha = \begin{cases} 1 & \text{if } d_{\mathbb{G}(n,n,p)}(\alpha) \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathbb{E}I_\alpha = np(1-p)^{n-1} + (1-p)^n$. Let $X = \sum_{\alpha \in A} I_\alpha$. Then $\mathbb{E}X = 2n(\binom{n}{1}p(1-p)^{n-1} + (1-p)^n) \sim 2n^2p(1-p)^{n-1}$. Let α and α' belong to different partitions of $\mathbb{G}(n,n,p)$. Then $cov(I_\alpha, I_{\alpha'}) = \mathbb{E}(I_\alpha I_{\alpha'}) - \mathbb{E}I_\alpha\ \mathbb{E}I_{\alpha'} \backsim n^2p^3(1-p)^{2n-3}$. Note that $cov(I_\alpha, I_\alpha) = \mathbb{E}I_\alpha^2 - (\mathbb{E}I_\alpha)^2$, hence $\mathrm{Var}X = 2n^2\ cov(I_\alpha, I_{\alpha'}) + 2n\ cov(I_\alpha, I_\alpha) \backsim 2n^4p^3(1-p)^{2n-3} + 2n^2p(1-p)^{n-1}$.

If $c_n \longrightarrow -\infty$, then $\mathbb{E}X \to \infty$ and hence

$$\mathbb{P}(X = 0) \leq \frac{VarX}{(\mathbb{E}X)^2} \backsim p/2 + 1/\mathbb{E}X \to 0.$$

Hence the Second Moment Method implies that the random bipartite graph has at least one leaves, which makes matching-covering impossible.

If $c_n \longrightarrow c$, then $\lambda = \mathbb{E}X = 2n^2(1-p)^{n-1}p \sim 2n^2pe^{-np} \to 2e^{-c}$ and we have

$$\frac{\mathrm{Var}X}{\mathbb{E}X} \backsim n^2p^2(1-p)^n + 1 = 1 + o(1).$$

Note that for a fixed α, the values of I_α is non-increase when the number of edges in $\mathbb{G}(n,n,p)$ increases. We apply Lemma 8 to Y_j, where Y_j is the edge indicator in the complement of $\mathbb{G}(n,n,p)$. It follows that the variables $\{I_\alpha\}_{\alpha \in A}$ are positively related. By Lemma 7, we have $X \xrightarrow{d} Po(2e^{-c})$. In particular,

$$\mathbb{P}(\mathbb{G}(n,n,p) \text{ has no leaves}) = \mathbb{P}(X = 0) \to e^{-2e^{-c}}.$$

Hence, the probability that $\mathbb{G}(n,n,p)$ is matching-covered is $e^{-2e^{-c}}$.

If $c_n \longrightarrow \infty$, then

$$\mathbb{E}X = 2n^2(1-p)^{n-1}p + 2n(1-p)^n \leq 3n^2pe^{-np} = o(1)$$

Applying the First Moment Method, we obtain, *a.a.s.* $\mathbb{G}(n,n,p)$ has no leaf. Thus a.a.s. $\delta(\mathbb{G}(n,n,p)) \geq 2$ and $\mathbb{G}(n,n,p)$ is matching-covered by Lemma 3. $\square$

4 Random Non-bipartite Matching-Covered Graphs

It was known that bicritical graphs are matching-covered. Next, we find the threshold of bicritical graph, and through it, we obtain a probability sequence of non-bipartite matching-covered graphs, which acts the same way like a threshold.

Lemma 11. (Bollobás, [2]) *Let $\omega(n) \leq \log\log\log n$, $\omega(n) \to \infty$ and $-\omega(n) \leq np - \log n - \log\log n \leq \omega(n)$. Let c be a constant. Then a.a.s. no two vertices of degree at most $\frac{1}{10}\log n$ are within distance c of each other.*

By Lemma 11, we see that, almost every vertex with finite degree is not contained in finite cycle. To obtain the threshold of bicritical graphs, we need to show the next theorem, which requires several results from [3, 8] as lemmas.

Theorem 3. *Suppose* $-\omega(n) \le np - \log n - 2\log\log n \le \omega(n)$, *where* $\omega(n) \le \log\log\log n$ *and* $\omega(n) \to \infty$, *and* $\lim_{n\to\infty} \mathbb{P}\{\delta(\mathbb{G}(n,p)) \ge 3\} > 0$. *If* $\delta(\mathbb{G}(n,p)) \ge 3$, *then a.a.s.* $\mathbb{G}(n,p)$ *is bicritical.*

Proof. If $\delta(\mathbb{G}(n,p)) \ge 3$, then by Lemma 10, $\mathbb{G}(n,p)$ is 3-connected. The combinatorial basis of the proof is the extension of the proof of Lemma 1. If a graph G is not bicritical, by Lemma 1, there exists a subset $R \subseteq V(G)$ such that $q(G-R) \ge |R|$.

For $r = 2, 3, \ldots$, denote by B_r the event that there is a minimal set $R \subset V$ such that $|R| = r$, and $\mathbb{G}(n,p) - R$ has at least r components, and each of which is either an isolated vertex or has at least three vertices. Define $B(r,s)$ the event that B_r holds and the union of $r-1$ smallest components in the previous statement, which we denote by S, with s elements. If $s \le r-2$, then $B_r = \emptyset$.

If $\mathbb{G}(n,p)$ does not satisfy the conclusions of the theorem, then B_r holds for some r, $0 \le r \le n/2$. Hence, the theorem is proved if we show

$$\mathbb{P}\left(\bigcup_{r=1}^{\lfloor n/2 \rfloor} B_r\right) = o(1)$$

Let $r_0 = (4\log\log n/\log n)n$, and A_1 be the event that $\mathbb{G}(n,p)$ does not have r_0 independent vertices.

Fact 1. ([3]) $\mathbb{P}(A_1) = 1 + o(1)$.

Now, we only need to show that

$$\mathbb{P}\left(\bigcup_{r=1}^{r_0} B_r\right) = o(1).$$

Let A_2 be the event that the complement of $\mathbb{G}(n,p)$ contains no K_{r_0,r_0}, i.e., a complete bipartite graph with r_0 vertices in each bipartition.

Fact 2. ([3]) $\mathbb{P}(A_2) = 1 + o(1)$.

Fact 3. ([3]) Let $s_0 = n^{1/2}\log^3 n$. Then

$$\mathbb{P}\left(\bigcup_{r=s_0}^{r_0} B(r) \cup \bigcup_{r=2}^{s_0} \bigcup_{s=s_0}^{r_0} B(r,s)\right) = o(1).$$

Fact 4. ([8]) Let E_l denote the set of those graphs which contain a subset S of l vertices which are connected by $\le l-1$ edges with vertices outside S. Then we have

$$\sum_{1 \le l \le \frac{n}{2\log n}} \mathbb{P}(E_l) = o(1)$$

Hence by Fact 3, we only need to show that

$$\mathbb{P}\left(\bigcup_{r=2}^{s_0}\bigcup_{s=r-1}^{s_0} B(r,s)\right) = o(1).$$

We have reduced the original problem now to the investigation of graphs having $r < n^{1/2}\log^3 n$ separating vertices such that after removing these vertices, the remaining graph contains $r-1$ components with $a_1, \ldots, a_{r-1}$ vertices, such that $s = a_1 + \ldots + a_{r-1} < n^{1/2}\log^3 n$. Let us denote the number of edges connecting one of the separating vertices with one of the s vertices belonging to the $r-1$ components by l. We consider the following two cases.

Case 1. $l \geq r+8$.

By Lemmas 10 and 11, we may assume $r \geq 3$. The probability of such a configuration clearly does not exceed

$$\begin{aligned}
\triangle &= \sum_{r=3}^{\sqrt{n}\log^3 n}\sum_{s=r-1}^{\sqrt{n}\log^3 n}\binom{n}{r}\binom{n-r}{s}\sum_{l=r+8}^{sr}\binom{sr}{l}p^l(1-p)^{s(n-s)-l}\\
&\leq \sum_{r=3}^{\sqrt{n}\log^3 n}\sum_{s=r-1}^{\sqrt{n}\log^3 n}\frac{n^{s+r}}{s!r!}(1-p)^{s(n-s)}\sum_{l=r+8}^{sr}\binom{sr}{l}\left(\frac{p}{1-p}\right)^l\\
&\leq \sum_{r=3}^{\sqrt{n}\log^3 n}\sum_{s=r-1}^{\sqrt{n}\log^3 n}\frac{n^{s+r}}{s!r!}(1-p)^{s(n-s)}\left(\frac{srpe}{r+8}\right)^{r+8}\\
&\leq \frac{1}{n^{-8}}\sum_{r=3}^{\sqrt{n}\log^3 n}\sum_{s=r-1}^{\sqrt{n}\log^3 n}\frac{1}{s!r!}\left(\frac{sre\log n}{r+8}\right)^{r+8}\\
&\leq \frac{c}{n^{-8}}\sum_{r=3}^{\sqrt{n}\log^3 n}\frac{1}{(r-1)!r!}\left(\frac{(r-1)re\log n}{r+8}\right)^{r+8}\\
&= O(n^{e^2-8}\log^m n) = o(1),
\end{aligned}$$

where c and m are constants.

Case 2. $l < r+8$.

Our original problem has been transformed to the study of graphs $\mathbb{G}(n,p)$ which satisfying the following conditions.

(i) select r separating vertices in $\mathbb{G}(n,p)$ with $3 \leq r < n^{1/2}\log^3 n$, removing these vertices, the graph $\mathbb{G}(n,p)$ falls into components, among which there are $r-1$ components $C_1, \ldots, C_{r-1}$ of orders $a_1, \ldots, a_{r-1}$, respectively;

(ii) there are l edges in $\mathbb{G}(n,p)$ connecting a separating vertex with a vertex in one of the components $C_1, \ldots, C_{r-1}$, where $3(r-1) \leq l \leq r+7$;

(iii) letting $s = a_1 + \cdots + a_{r-1}$, by Fact 4, we have $s \leq l$.

By (ii), we obtain $r \leq 7$. From Lemma 11, we have $|a_1| = \ldots = |a_{r-1}| = 1$. Then the degree of vertices in $C_1 \cup \cdots \cup C_{r-1}$ is at most 7, a contradiction to Lemma 11.

Thus we have shown that the threshold for being bicritical coincides with that for the disappearance of degree 2, which proves Theorem 3. □

Theorem 4. *Let $c_n = np - \log n - 2\log\log n$. Let $n \equiv 0 \pmod 2$. Then*

$$\mathbb{P}(\mathbb{G}(n,p) \textit{ is bicritical}) \longrightarrow \begin{cases} 0 & \textit{if } c_n \to -\infty, \\ e^{-e^{-c}/2} & \textit{if } c_n \to c, \\ 1 & \textit{if } c_n \to \infty. \end{cases}$$

Proof. We consider three cases.

Case 1. $c_n \to -\infty$.

Then a.a.s. $\delta(\mathbb{G}(n,p))$ is less than 3, by Lemma 2 and Theorem 3, a.a.s. $\mathbb{G}(n,p)$ is not bicritical.

Case 2. $c_n \to \infty$.

Then a.a.s. $\delta(\mathbb{G}(n,p))$ is not less than 3, and hence by Theorem 3, a.a.s. $\mathbb{G}(n,p)$ is bicritical.

Case 3. $c_n \to c$.

By Lemma 10, we have $\mathbb{P}\{\delta(\mathbb{G}(n,p)) = \kappa(\mathbb{G}(n,p)) \geq 3\} = e^{-e^{-c}/2}$, hence by Theorem 3, $\mathbb{P}\{\mathbb{G}(n,p)$ *is* bicritical$\} = e^{-e^{-c}/2}$. □

With a similar argument, we can see that a.a.s. all 3-connected graphs are bricks.

Theorem 5. *Let $c_n = np - \log n - \log\log n$. Let $n \equiv 0 \pmod 2$. Then*

$$\mathbb{P}(\mathbb{G}(n,p) \textit{ is matching-covered}) \longrightarrow \begin{cases} 0 & \textit{if } c_n \to -\infty, \\ e^{-e^{-c}} & \textit{if } c_n \to c, \\ 1 & \textit{if } c_n \to \infty. \end{cases}$$

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