

# On Some Properties of Cages\*

Tao Wang<sup>1</sup>, Baogang Xu<sup>2</sup> and Qinglin Yu<sup>3†</sup>

<sup>1</sup>School of Mathematics and Information Sciences  
Henan University, Kaifeng, China

<sup>2</sup>School of Mathematics and Computer Science  
Nanjing Normal University, Nanjing, China

<sup>3</sup>Department of Mathematics and Statistics  
Thompson Rivers University, Kamloops, BC, Canada

## Abstract

A  $(k; g)$ -cage is a graph with the minimum order among all  $k$ -regular graphs with girth  $g$ . As a special family of graphs,  $(k; g)$ -cages have a number of interesting properties. In this paper, we investigate various properties of cages, e.g., connectivity, the density of shortest cycles, bricks and braces.

*Key words:* cage, diameter, girth, connectivity, brick.

## 1 Introduction

A  $(k; g)$ -graph is a  $k$ -regular graph with girth  $g$ . Let  $f(k; g)$  be the smallest integer  $v$  such that there exists a  $(k; g)$ -graph with  $v$  vertices. A  $(k; g)$ -cage is a  $(k; g)$ -graph with  $f(k; g)$  vertices. Cages have been intensely studied since its introduction by Tutte in [14]. However, most work has focused on the existence problem and little is known for the structural properties of cages.

Recently, several authors have studied structural properties of cages, e.g., degree monotonicity [16], the factorial property [9] and separating properties [4], but many results are in the area of connectivity. Fu et al. [2] proved that all  $(k; g)$ -cages are 2-connected. They further conjectured that every  $(k; g)$ -cage is  $k$ -connected and confirmed this conjecture for the case  $k = 3$ . In [1, 4], it has been proved that all  $(k; g)$ -cages with  $k \geq 3$  are 3-connected. In addition,  $(4; g)$ -cages are 4-connected (see [17]). Most recently, it was proved that when  $k \geq 4$  and  $g \geq 10$ , all  $(k; g)$ -cages are 4-connected (see [12]); and  $r$ -connected for  $r \geq \sqrt{k+1}$

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†Corresponding email: yu@tru.ca

whenever  $k$  is odd (see [6]); and for the even case,  $r$  is the largest integer such that  $(r-1)^3 + 2(r-1)^2 \leq k$  (see [5]). For the edge-connectivity of  $(k;g)$ -cages, it was shown that all  $(k;g)$ -cages are  $k$ -edge connected through some collective effort (see [8, 15]).

Moreover, several papers have focused on refined concepts of connectivity, such as quasi-connectivity and edge-super-connectivity [7, 11, 13]. A graph is *edge-super-connected* if all its minimum edge-cuts are sets of edges incident with one vertex.

We use  $\delta(G)$  for the minimum degree of  $G$ ,  $\deg_G(v)$  for the degree of  $v$  in  $G$ , and  $d_G(u, v)$  for the distance between  $u$  and  $v$  in  $G$ , respectively. Let  $N_H(v)$  denote the neighborhood of  $v$  in  $H$ ,  $\text{diam}(G)$  denotes the diameter of  $G$ .

In [4], Jiang and Mubayi studied the separating sets of cages, such as stars and cycles, and obtained the following general theorem about vertex cuts in cages.

**Theorem 1.1** (Jiang and Mubayi, [4]). *Suppose that  $G$  is a  $(k;g)$ -cage and  $S$  is a vertex cut of  $G$ . Then  $\text{diam}(G[S]) \geq \lfloor \frac{g}{2} \rfloor$ . Furthermore, the inequality is strict if  $d_{G[S]}(u, v)$  is maximized for exactly one pair of vertices  $u$  and  $v$ .*

As a consequence of the above theorem,  $\text{diam}(G[S]) \leq \lfloor \frac{g}{2} \rfloor - 1$  implies that  $G - S$  is connected. In Section 2, we further show that if  $\text{diam}(G[S]) \leq \lfloor \frac{g}{2} \rfloor - 2$ , then  $G - S$  is 2-connected.

In Section 3, we consider the density of small cycles (cycles with length  $g$  or  $g+1$ ). In a  $(k;g)$ -cage, we call the cycles of length  $g$  as *girth cycles*. We are interested in the distribution of girth cycles in cages. In particular, does there exist a girth cycle through any given vertex or edge? In this section, we prove that in a  $(k;g)$ -cage, each vertex and each edge is contained in a cycle of length at most  $g+1$ . Furthermore, we also present a necessary condition for the existence of the girth cycles of cages.

In Section 4, we prove some properties about bricks and braces for cages.

Throughout this paper, we use several times the Girth Monotonicity Theorem, which is an important property established by numerous researchers independently.

**Theorem 1.2** (Girth Monotonicity Theorem). *If  $k \geq 3$  and  $3 \leq g_1 < g_2$ , then  $f(k;g_1) < f(k;g_2)$ .*

## 2 A Connectivity Property

To show that each  $(k;g)$ -cage is 4-connected, Marcote et al. [12] established the following technical lemma, which will be used in our proof as well.

**Lemma 1.** *Let  $G$  be a  $k$ -regular graph with  $k \geq 3$  and girth  $g \geq 5$ . Let  $H$  be a subgraph of  $G$  with the minimum degree  $k-1$ . Let  $\Omega = \{v \in V(H) \mid \deg_H(v) = k-$*

*1). Suppose that  $\Omega$  can be partitioned into two sets  $\Omega_1$  and  $\Omega_2$  with  $|\Omega_2| = m \geq 2$ , in such a way that*

(a)  $d_H(u_i, v_j) \geq \lfloor (g-1)/2 \rfloor$  for every pair of distinct vertices  $u_i, v_j \in \Omega_1$ ;

(b)  $\Omega_2 = \{z_1, z_2, \dots, z_m\} \subseteq N(z)$  for some vertex  $z \notin V(H)$ .

*Then,  $f(k;g) \leq 2|V(H)|$  if either  $|\Omega_1| < m$  or  $2m + 2 \sum_{i=1}^m d_H(u_i, z_i) \geq g$  for any  $\{u_1, u_2, \dots, u_m\} \subseteq \Omega_1$ .*

**Theorem 2.1.** *Let  $G$  be a  $(k;g)$ -cage with  $k \geq 3$  and  $g \geq 7$ . For any  $S \subseteq V(G)$ , if  $\text{diam}(G[S]) \leq \lfloor g/2 \rfloor - 2$ , then  $G - S$  is 2-connected.*

*Proof.* Suppose, to the contrary, that  $G - S$  has a cut vertex. We choose a cut vertex  $v$  of  $G - S$  to minimize the order of the smallest component of  $G - S - v$ . Let  $H$  be the smallest component of  $G - S - v$ , and let  $S' = S \cup \{v\}$ . Clearly,  $H$  is not a singleton and  $|N_H(v)| \geq 2$  (otherwise, we can choose the only neighbor of  $v$  in  $H$  as the cut vertex of  $G - S$ ).

It is not hard to see that every vertex in  $H$  has at most one neighbor in  $S$  and thus  $\delta(H) \geq k - 2$ .

*Case 1.*  $\delta(H) = k - 2$ .

Clearly, there is only one vertex  $w \in V(H)$  such that  $\deg_H(w) = k - 2$ . Otherwise, there is a cycle with length at most  $4 + (\lfloor g/2 \rfloor - 2)$ , which is less than  $g$  since  $g \geq 7$ , a contradiction. Let  $H' = H - w$ .

*Case 1.1.*  $|N_H(v)| \geq 3$ , i.e.,  $|N_{H'}(v)| \geq 2$ .

In this case,  $\delta(H') = k - 1$ . Let  $\Omega_1 = N_{H'}(S) \cup N_{H'}(w)$ ,  $\Omega_2 = N_{H'}(v) = \{z_1, z_2, \dots, z_m\}$ , where  $|\Omega_2| = m \geq 2$  (see Fig. 1(a)).

For any vertices  $x, y \in \Omega_1$ , if  $x, y \in N_{H'}(S)$ , then  $d_{H'}(x, y) \geq g - 2 - (\lfloor \frac{g}{2} \rfloor - 2) = \lfloor \frac{g}{2} \rfloor > \lfloor \frac{g-1}{2} \rfloor$ ; if  $x, y \in N_{H'}(w)$ , then  $d_{H'}(x, y) \geq g - 2 > \lfloor \frac{g-1}{2} \rfloor$ ; if  $x \in N_{H'}(S)$  and  $y \in N_{H'}(w)$ , then  $d_{H'}(x, y) \geq g - 3 - (\lfloor \frac{g}{2} \rfloor - 2) = \lfloor \frac{g}{2} \rfloor - 1 = \lfloor \frac{g-1}{2} \rfloor$ . Therefore,  $d_{H'}(x, y) \geq \lfloor \frac{g-1}{2} \rfloor$  for any  $x, y \in \Omega_1$ .

Suppose  $|\Omega_1| \geq m$ . For every  $\{u_1, u_2, \dots, u_m\} \subseteq \Omega_1$ , if there is no vertex  $u_i \in N_{H'}(w)$ , then  $2m + 2 \sum_{i=1}^m d_{H'}(u_i, z_i) \geq 4 + 2(g - 4 - (\lfloor g/2 \rfloor - 2)) = 2\lfloor g/2 \rfloor \geq g$ .

If some  $u_i \in N_{H'}(w)$ , then  $2m + 2 \sum_{i=1}^m d_{H-w}(u_i, z_i) \geq 4 + 2(g - 3 + 1) = 2g > g$ . By Lemma 1, we have  $|V(H')| \geq |V(G)|/2$ , a contradiction.

*Case 1.2.*  $|N_H(v)| = 2$ , i.e.,  $|N_{H'}(v)| = 1$ .

Let  $z$  be the other neighbor of  $v$  in  $H$ ,  $H'' = H' - z$  and  $N_{H''}(z) = \{z_1, z_2, \dots, z_{k-1}\}$ . Since  $g \geq 7$ , then  $\delta(H'') = k - 1$ . Let  $\Omega_1 = N_{H''}(S) \cup N_{H''}(w)$ ,  $\Omega_2 = N_{H''}(z) = \{z_1, z_2, \dots, z_{k-1}\}$ , so  $|\Omega_2| = k - 1 \geq 2$  (see Fig. 1(b)).



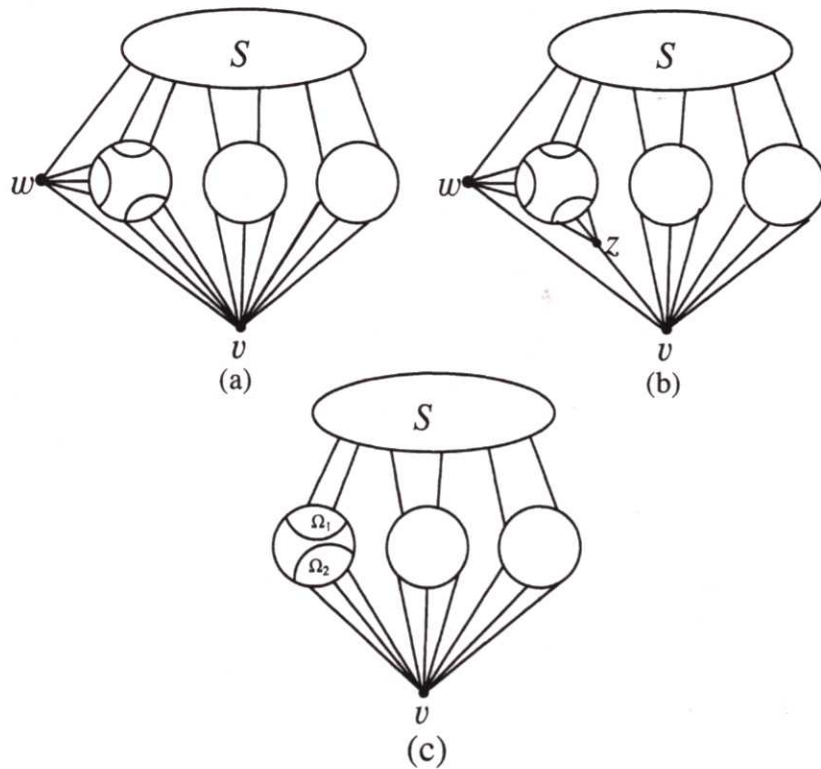


Figure 1:  $\delta(H) = k - 2$

Similar to Case 1.1, for any vertices  $x, y \in \Omega_1$ ,  $d_{H''}(x, y) \geq \lfloor \frac{g-1}{2} \rfloor$ . Suppose  $|\Omega_1| \geq k - 1$ , for every  $\{u_1, u_2, \dots, u_{k-1}\} \subseteq \Omega_1$ ,  $2(k-1) + 2 \sum_{i=1}^{k-1} d_{H''}(u_i, z_i) \geq g$ . By Lemma 1, we have  $|V(H'')| \geq |V(G)|/2$ , a contradiction.

Case 2.  $\delta(H) = k - 1$ .

Let  $\Omega_1 = N_H(S)$ ,  $\Omega_2 = N_H(v) = \{z_1, z_2, \dots, z_m\}$ ,  $|\Omega_2| = m \geq 2$  (see Fig. 1(c)).

For any two vertices  $x, y \in \Omega_1$ , we have  $d_H(x, y) \geq g - 2 - (\lfloor \frac{g}{2} \rfloor - 2) = \lfloor \frac{g}{2} \rfloor > \lfloor \frac{g-1}{2} \rfloor$ . If  $|\Omega_1| \geq m$ , then for every  $\{u_1, u_2, \dots, u_m\} \subseteq \Omega_1$ , it follows  $2m + 2 \sum_{i=1}^m d_H(u_i, z_i) \geq 4 + 2(g - 4 - (\lfloor g/2 \rfloor - 2)) = 2\lceil g/2 \rceil \geq g$ . By Lemma 1, we have  $|V(H)| \geq |V(G)|/2$ , a contradiction.

Thus, we complete the proof of the theorem.  $\square$

It was proved in [1] that a  $(k; g)$ -cage contains no star vertex cut if  $g \geq 6$ , that

is, for any vertex  $v$ ,  $G - (N(v) \cup \{v\})$  is still connected. Theorem 2.1 yields the following corollary directly.

**Corollary 1.** Let  $G$  be a  $(k; g)$ -cage and  $S = N_G(v) \cup \{v\}$ . Then  $G - S$  is 2-connected for  $k \geq 3$  and  $g \geq 8$ .

### 3 Girth cycles

In [3], Jiang proved the following.

**Theorem 3.1** (Jiang [3]). Let  $G$  be a  $(k; g)$ -cage. Then every edge of  $G$  is contained in at least  $k - 1$  cycles of length at most  $g + 1$ .

From this theorem, the following result follows immediately.

**Corollary 2.** Every vertex of a  $(k; g)$ -cage is contained in at least  $k - 1$  cycles of length at most  $g + 1$ .

However, for even  $k$ , the distribution of girth cycles is better understood as shown in the next theorem, which is well known among researchers but not appear formally in any literature yet. It was implicitly proved in Theorem 1 of [2].

**Theorem 3.2.** Let  $G$  be a  $(k; g)$ -cage. If  $k$  is even, then every vertex of  $G$  is contained in a girth cycle.

In the case of odd  $k$ , we prove the following result.

**Theorem 3.3.** Let  $G$  be a  $(k; g)$ -cage and  $k$  be odd. Let  $\mathcal{V} = \{v \in V(G) \mid \text{the shortest cycle through } v \text{ is of length } g + 1\}$ . Then for any two vertices  $x, y \in \mathcal{V}$ ,  $d_G(x, y) \geq 4$ .

*Proof.* Suppose that there exists two vertices  $x, y \in \mathcal{V}$ , such that  $d_G(x, y) < 4$ . We construct a  $k$ -regular graph  $G'$  by deleting vertices  $x$  and  $y$  and adding some edges (see Fig. 2).

When  $d_G(x, y) = 1$ , delete the vertices  $x, y$  and add a perfect matching in  $N_G(x) \setminus \{y\}$  and  $N_G(y) \setminus \{x\}$  (see Fig. 2(a)).

When  $d_G(x, y) = 2$ , let  $xzy$  be the shortest path between  $x$  and  $y$ , delete the vertices  $x$  and  $y$ , and add a perfect matching in  $N_G(y) \setminus \{z\}$ , and add two edges joining  $z$  to  $N_G(x)$ , and a perfect matching of the remaining vertices in  $N_G(x)$  (see Fig. 2(b)).

When  $d_G(x, y) = 3$ , let  $xzz'y$  be the shortest path between  $x$  and  $y$ , delete the vertices  $x, y$  and the edge  $zz'$ , add two edges joining  $z$  to  $N_G(x)$  and two edges joining  $z'$  to  $N_G(y)$ , and then add two perfect matchings in the remaining vertices in  $N_G(x)$  and  $N_G(y)$  (see Fig. 2(c)).

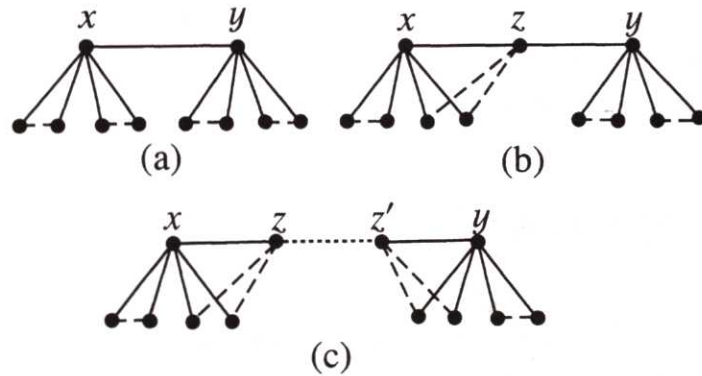


Figure 2: The local structure of the graph

Next, we show that the girth of  $G'$  is at least  $g$ . Let  $C$  be a girth cycle of  $G'$ . If  $C$  is a cycle of  $G$ , then  $|C| \geq g$ ; if  $G'$  contains only one new edge  $e = uv$ , then  $C - e$  is a path with two ends both in  $N_G(x)$  or  $N_G(y)$  and thus  $|C - e| \geq g - 1$  or  $|C| \geq g$ ; if  $C$  contains at least two new independent edges, then  $|C| \geq 2(g - 4) + 2 \geq g$ . Hence  $G'$  is a  $k$ -regular graph with girth at least  $g$ , but  $|V(G')| < |V(G)|$ , a contradiction.  $\square$

## 4 Bricks and braces

A nontrivial connected graph is called *matching covered* if every edge is contained in a perfect matching. A graph  $G$  is *bicritical* if for any two distinct vertices  $u$  and  $v$ ,  $G - \{u, v\}$  has a perfect matching. A *brick* is a 3-connected bicritical graph. A bipartite matching covered graph  $G$  with bipartite  $(A, B)$  is called *brace*, if  $(A - \{x\}, B - \{y\})$  has a perfect matching for every  $x \in X$  and  $y \in B$ . Bricks and braces have played a prominent roles in the brick decomposition and the study of matching lattices (see [10]). Their properties and classification have attracted a great interest.

Let  $G$  be a graph and  $\nabla(S)$  denote edge set of  $G$  which has precisely one end in  $S$ . Let  $G$  be a matching covered graph, a cut  $K = \nabla(S)$  is a *tight cut* if for every perfect matching  $M$  of  $G$ ,  $|M \cap K| = 1$ . Clearly,  $\nabla(v)$  is a tight cut for any vertex  $v$ , such a cut is called *trivial*.

Next we show that all cages with even order are either bricks or braces.

**Lemma 2** (Lovász [10]). *A matching covered graph has no nontrivial tight cut if and only if it is either a brick or a brace.*

**Lemma 3** (Lin et al.[7]). *All  $(k; g)$ -cages are edge-super-connected.*

**Lemma 4** (see [1, 4]). *If  $k \geq 3$  and  $G$  is a  $(k; g)$ -cage, then  $G$  is 3-connected.*

The next is the main result of this section.

**Theorem 4.1.** *All  $(k; g)$ -cages with  $k \geq 3$  of even order are bricks or braces.*

*Proof.* Suppose that a  $(k; g)$ -cage  $G$  is not bicritical.

*Claim 1.*  $G$  is a bipartite graph.

If  $G$  is not bicritical, then there exists a vertex set  $S \subset V(G)$  with  $|S| \geq 2$  such that  $c_o(G - S) > |S| - 2$ , where  $c_o(G - S)$  denote the number of odd components of  $G - S$ . By parity,  $c_o(G - S) \geq |S|$ . Since  $G$  is  $k$ -regular, there are at most  $k|S|$  edges going out of  $S$ . Since  $G$  is  $k$ -edge-connected, then there are at least  $kc(G - S)$  edges going out of  $G - S$ . Hence,  $kc(G - S) \leq E(S, V - S) \leq k|S|$ , and  $c(G - S) = |S|$ , and this implies that  $S$  is an independent set and every component of  $G - S$  has exactly  $k$  edges going out. Moreover,  $G$  is edge-superconnected, thus every component of  $G$  is an isolated vertex. Therefore,  $G$  is a  $k$ -regular bipartite graph. By Hall's Theorem,  $G$  is matching covered graph.

*Claim 2.*  $G$  has no nontrivial tight cut.

Otherwise,  $G$  has a nontrivial tight cut  $K = \nabla(X)$ . Since  $G$  is edge-superconnected, then  $|K| > k$ . We contract  $X$  to a vertex  $x$  to obtain a simple graph  $H$ . Since  $K$  is a tight cut, it is easy to verify that  $H$  is a matching covered graph. By Claim 1,  $H$  is a bipartite graph with bipartition  $(U, W)$ . Since  $H$  is matching-covered, so  $H$  of course has perfect matching and  $|U| = |W|$ . In  $H$ , since  $x$  is the only new vertex, so other vertices are all in  $G$  and have degree  $k$ . Since  $|U| = |W|$ , so  $x$  must have degree  $k$  too, i.e.  $|K| = k$ , a contradiction. Therefore,  $G$  has no nontrivial tight cut, by Lemma 2,  $G$  is a brace.

By Lemma 4, the theorem follows.  $\square$

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