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# Edge Disjoint Hamilton Cycles in Intersection Graphs of Bases of Matroids *

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**Abstract**

The *intersection graph* for bases of a matroid $M = (E, \mathcal{B})$ is a graph $G^I(M)$ with vertex set $\mathcal{B}$ and edge set $\{BB' : |B \cap B'| \neq 0, B, B' \in \mathcal{B}\}$. In this paper, we prove that the intersection graph $G^I(M)$ for bases of a simple matroid $M$ with rank $r(M) \geq 2$ has at least two edge-disjoint Hamilton cycles whenever $|V(G^I(M))| \geq 5$.

*Keywords*: Matroid, intersection graph, base, Hamilton cycle.

## 1 Introduction

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A *matroid* $M = (E, \mathcal{B})$ is a finite set $E$ together with a nonempty collection $\mathcal{B}$ of subsets of $E$ that satisfies the following condition: for any $B, B' \in \mathcal{B}$ with $|B| = |B'|$ and for any $e \in B \setminus B'$, there exists $e' \in B' \setminus B$ such that $(B \setminus \{e\}) \cup \{e'\} \in \mathcal{B}$. Each member of $\mathcal{B}$ is called a *base* of $M$. An element of $E$ that is contained in every base is called a *coloop*, and an element of $E$ that is contained in no base is called a *loop*. A matroid without loops and 2-circuits is called a *simple* matroid. The *rank* $r$ of a matroid is the number of elements in a base. We denote the uniform matroid of rank $m$ on an $n$-element set by $U_{m,n}$.

The *base graph* of a matroid $M = (E, \mathcal{B})$ is the graph $G' = G'(M)$ with vertex set $V(G') = \mathcal{B}$ and edge set $E(G') = \{BB' : B, B' \in \mathcal{B}$ and $|B \setminus B'| = 1\}$, where the same notation is used for the vertices of $G'$ and the bases of $M$. The basic properties and characterizations of base graphs of matroids can be found in [9].

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Next, we extend base graphs into a family of larger graphs by relaxing the requirement for vertices adjacency as follows: The *intersection graph* for bases of a matroid $M = (E, \mathcal{B})$ is the graph, denoted by $G^I(M)$, with vertex set $V(G^I) = \mathcal{B}$ and edge set $E(G^I) = \{BB' : |B \cap B'| \neq 0, B, B' \in \mathcal{B}(M)\}$.

For $r(M) = 1$, the intersection of any two bases of $M$ is empty and thus we see that the intersection graph $G^I(M)$ with rank $r(M) = 1$ is a collection of $|\mathcal{B}|$ isolated vertices. Clearly, for $r(M) \geq 2$, the intersection graph $G^I(M)$contains the base graph $G'(M)$ as a connected spanning subgraph. In particular, for $r(M) = 2$, the intersection graph $G^I(M)$ is exactly the base graph $G'$ of $M$. The intersection graph $G^I$ for bases of matroid $U_{2,4}$ is shown in Fig.1.

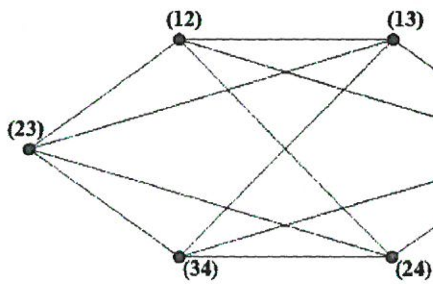


Fig. 1

The problem of Hamilton cycles in base graphs of matroids have been investigated by many researchers. Cummins [5] showed that the base graph of a matroid with at least three vertices has a Hamilton cycle. Bondy [3] showed not only that every base graph is Hamiltonian, but also that most are pancyclic. Holzmann and Harary [6] showed that for every edge in the base graph of a matroid there is a Hamilton cycle containing it and another Hamilton cycle avoiding it.

The existence of disjoint Hamilton cycles in graphs in general is a very challenging problem, only limited knowledge exists in the literature. Most known results are involved with either large degree sum or large connectivity as a sufficient condition (e.g., [7], [8]). Since the base graph $G'(M)$ contains Hamilton cycles and the intersection graph $G^I(M)$ contains the base graph $G'(M)$ as a connected spanning subgraph, it is natural to explore the problem of disjoint Hamilton cycles in the intersection graph $G^I$. In this paper, we prove that there are at least two edge-disjoint Hamilton cycles in the intersection graph $G^I(M)$ for bases of any simple matroid $M$ whenever $|V(G^I)| \geq 5$. Note that if a matroid $M$ contains a coloop $e$, then every base of $M$ contains $e$. Thus the intersection graph $G^I(M)$ is a complete graph. Clearly in this case there are at least two edge-disjoint Hamilton cycles in the intersection graph $G^I$ whenever $|V(G^I)| \geq 5$.

For $v \in V(G), A \subseteq V(G), B \subseteq V(G) - A$, we define $N_A(v;G) = \{x \in A, vx \in E(G)\}, N_A(B;G) = \bigcup_{y \in B} N_A(y;G)$, and $E_G(A,B) = \{xy \in E(G) : x \in A, y \in B\}$.

Terminology and notations not defined here can be found in [10].

# 2 Main result and its proofs

The main result of this paper is the following.

**Theorem 2.1.** *Let $M = (E, \mathcal{B})$ be a simple matroid with rank $r(M) \geq 2$ and $G^I$ be the intersection graph for bases of $M$. If $|V(G^I)| \geq 5$, then $G^I$ has at least two edge-disjoint Hamilton cycles.*

To prove this theorem, we start with a few well-known results.

**Lemma 2.2.** (see [2]) *The complete graph $K_n (n \geq 2k+1)$ has $k$ edge-disjoint Hamilton cycles. $K_n (n \geq 2k)$ has $k$ edge-disjoint Hamilton paths having any given $k$ pairs of vertices, which are mutually disjoint, as their end vertices.*

**Lemma 2.3.** ([6]) *Let $M = (E, \mathcal{B})$ be a matroid on $E$ and $e \in E$. If $G'$, $G'_1$ and $G'_2$ are the matroid base graphs of $M$, $M \backslash e$ and $M/e$, then $V(G'_1)$ and $V(G'_2)$ partition $V(G')$.*

**Lemma 2.4.** (Hall's Theorem, see [4]) *Let $G$ be a bipartite graph with bipartition $(X, Y)$. Then $G$ contains a matching that covers every vertex in $X$ if and only if $|N_Y(S;G)| \geq |S|$ for all $S \subseteq X$.*

Hereafter, we always assume that any matroid $M$ has no coloops, $r(M) \geq 2$ and $|\mathcal{B}| \geq 3$. In order to proceed to the proof of Theorem 2.1, we need several technical lemmas.

**Lemma 2.5.** (see [1]) *Let $G$ be a simple graph with two edge-disjoint Hamilton cycles $C_1$ and $C_2$. If $|V(G)| \geq 5$, then we can choose $e_1 \in C_1$ and $e_2 \in C_2$ such that $\{e_1, e_2\}$ is a matching of $G$.*

For any $e \in E \setminus B$, $B \cup \{e\}$ contains a unique basic circuit, denoted by $C(e, B)$, and we use $\mathcal{B}_e$ and $\overline{\mathcal{B}_e}$ to denote the bases containing $e$ and avoiding $e$, respectively.

**Lemma 2.6.** *Let $M = (E, \mathcal{B})$ be a simple matroid. If $|E| = n$ and $r(M) = r$, then $|\overline{\mathcal{B}_e}| \geq 2(n-r)-1$ and $|\mathcal{B}_e| \geq n-r+1$ for any $e \in E$.*

**Proof:** For any $e \in E$, since $M$ does not has loops or coloops, there exist bases $B_1$ and $B_2$ such that $e \notin B_1$ and $e \in B_2$. For any element $f \in (E \setminus B_1) \setminus \{e\}$, $C(f, B_1) \subseteq B_1 \cup \{f\}$ is a basic circuit of $M$ with respect to the base $B_1$. So there exist two elements $\{g_1, g_2\} \subseteq C(f, B_1) \setminus \{f\}$ such that $(B_1 \cup \{f\}) \setminus \{g_i\} \in \mathcal{B}(M)$ and $e \notin (B_1 \cup \{f\}) \setminus \{g_i\}$ $(i = 1, 2)$. So there are at least $2|(E \setminus B_1) \setminus \{e\}| + 1 = 2(n - r - 1) + 1 = 2(n - r) - 1$ bases avoiding $e$. Furthermore, for any element $x \in E \setminus B_2$, $C(x, B_2) \subseteq B_2 \cup \{x\}$ is a basic circuit of $M$ with respect to the base $B_2$. So there exists an element $y \in C(x, B_2) \setminus \{x, e\}$ such that $(B_2 \cup \{x\}) \setminus \{y\} \in \mathcal{B}(M)$ and $e \in (B_2 \cup \{x\}) \setminus \{y\}$. Hence there are at least $|E \setminus B_2| + 1 = n - r + 1$ bases of $M$ containing $e$. □

**Lemma 2.7.** *Let $M = (E, \mathcal{B})$ be a simple matroid on $E$ and $e$ be any element of $E$. Let $G'$, $G'_1$ and $G'_2$ be the base graphs of matroids $M$, $M \setminus e$ and $M/e$, respectively. If $r(M) = 2$ and $|E| = n \geq 5$ or $r(M) \geq 3$ and $|E| = n \geq 2r$, then for any four distinct vertices $B_1$, $B_2$, $B_3$ and $B_4$ of $V(G'_1)$, there exist four distinct vertices $B'_1$, $B'_2$, $B'_3$ and $B'_4$ of $V(G'_2)$ such that $B_1B'_1$, $B_2B'_2$, $B_3B'_3$ and $B_4B'_4$ are edges of $G'$.*

**Proof:** By Lemma 2.6, we have that $|V(G'_1)| \geq 5$ and $|V(G'_2)| \geq 4$. Let $\mathcal{B}_1 = \{B_1, B_2, B_3, B_4\}$ be any four vertices of $G'_1$. It is easy to see that $(B_i \cup \{e\}) \setminus \{e_i\} \in \mathcal{B}_e$ is a base of $M$ for any $e_i \in C(e, B_i) \setminus \{e\}$ and $B_iB'_i \in E(G')$ $(i = 1, 2, 3, 4)$. It is obvious that $|N_{G'_2}(B_i; G')| = |C(e, B_i) \setminus \{e\}| \geq 2$ $(i = 1, 2, 3, 4)$. Let $\mathcal{B}_2 = N_{G'_2}(\mathcal{B}_1; G') \subseteq V(G'_2)$. We consider the bipartite graph $H = (\mathcal{B}_1, \mathcal{B}_2)$ with vertex set $V(H) = \mathcal{B}_1 \cup \mathcal{B}_2$ and $E(H) = E_{G'}(\mathcal{B}_1; \mathcal{B}_2)$. Now we want to find four distinct vertices $\{B'_1, B'_2, B'_3, B'_4\} \subseteq \mathcal{B}_2$ such that $B_1B'_1$, $B_2B'_2$, $B_3B'_3$ and $B_4B'_4$ are edges of $G'$. It suffices to show that we can find a matching $N$ of $H$ covering every vertex of $\mathcal{B}_1$. By Lemma 2.4, we need only to check that for any subset $S$ of $\mathcal{B}_1$,

$$|N_{\mathcal{B}_2}(S; H)| \geq |S|. \qquad (*)$$

When $|S| = 1$, we have $|N_{\mathcal{B}_2}(S; H)| = |N_{\mathcal{B}_2}(B_i; H)| = |N_{G'_2}(B_i; G')| \geq |C(e, B_i) \setminus \{e\}| \geq 2 > 1 = |S|$ for each $i \in \{1, 2, 3, 4\}$. When $|S| = 2$, we have $|N_{\mathcal{B}_2}(S; H)| \geq |N_{\mathcal{B}_2}(B_i; H)| = |N_{G'_2}(B_i; G')| \geq 2 = |S|$ for any $B_i \in S$. Next we show that $(*)$ holds for $|S| = 3$ by contradiction. Suppose that there exists a subset $S$ of $\mathcal{B}_1$ with $|S| = 3$ such that $|N_{\mathcal{B}_2}(S; H)| < |S| = 3$. On the other hand, $|N_{\mathcal{B}_2}(S; H)| \geq |N_{G'_2}(B_i; G')| \geq 2$ for any $B_i \in S$. Thus we have $|N_{\mathcal{B}_2}(S; H)| = 2$. Without loss of generality, let $S = \{B_1, B_2, B_3\}$ and $N_{\mathcal{B}_2}(S; H) = \{B'_1, B'_2\} \subseteq \mathcal{B}_2$. Clearly, the subgraph of $H$ induced by $S \cup N_{\mathcal{B}_2}(S; H)$ is the complete bipartite graph $K_{3,2}$. This implies that there exists $\{e_i, e'_i\} \subseteq C(e, B_i) \setminus \{e\}$ $(e_i \neq e'_i)$ such that $(B_1 \cup \{e\}) \setminus \{e_1\} = (B_2 \cup \{e\}) \setminus \{e_2\} = (B_3 \cup \{e\}) \setminus \{e_3\} = B'_1$ and $(B_1 \cup \{e\}) \setminus \{e'_1\} =$

$(B_2 \cup \{e\}) \setminus \{e_2'\} = (B_3 \cup \{e\}) \setminus \{e_3'\} = B_2'$ for $i \in \{1,2,3,4\}$. It is easy to see that this is a contradiction. So we have $|N_{\mathcal{B}_2}(S;H)| \geq |S|$ for any subset $S$ of $\mathcal{B}_1$ when $|S| = 3$.

Finally, we show that $|N_{\mathcal{B}_2}(\mathcal{B}_1;H)| = |\mathcal{B}_2| \geq |\mathcal{B}_1| = 4$. If there exists $B' \in \mathcal{B}_2$ such that $d_H(B') = 3$. Without loss of generality, let $N_{\mathcal{B}_1}(B';H) = \{B_1, B_2, B_3\} \subseteq \mathcal{B}_1$. Then there exists $e_i \in C(e, B_i) \setminus \{e\}$ $(i = 1,2,3)$ such that $(B_1 \cup \{e\}) \setminus \{e_1\} = (B_2 \cup \{e\}) \setminus \{e_2\} = (B_3 \cup \{e\}) \setminus \{e_3\} = B'$. It is obvious that for any $e_i' \in C(e, B_i) \setminus \{e, e_i\}$ $(i = 1,2,3)$, we have $B_1' = (B_1 \cup \{e\}) \setminus \{e_1'\}$, $B_2' = (B_2 \cup \{e\}) \setminus \{e_2'\}$ and $B_3' = (B_3 \cup \{e\}) \setminus \{e_3'\}$. Furthermore, it is easy to see that $\{B_1', B_2', B_3', B'\}$ are four distinct bases of $M$. So $\{B_1', B_2', B_3', B'\} \subseteq \mathcal{B}_2$ and $|N_{\mathcal{B}_2}(\mathcal{B}_1;H)| = |\mathcal{B}_2| \geq 4 = |\mathcal{B}_1|$. If there exists $B' \in \mathcal{B}_2$ such that $d_H(B') = 4$, then we can prove that $|N_{\mathcal{B}_2}(\mathcal{B}_1;H)| = |\mathcal{B}_2| \geq 5 > |\mathcal{B}_1|$ similarly. Next we assume that $0 \leq d_H(B') \leq 2$ for any vertex $B'$ of $\mathcal{B}_2$. Since

$$\sum_{i=1}^{|\mathcal{B}_2|} d_H(B_i') = \sum_{j=1}^{4} d_H(B_j) \geq 8,$$

we have $|N_{\mathcal{B}_2}(\mathcal{B}_1;H)| = |\mathcal{B}_2| \geq 4 = |\mathcal{B}_1|$. So we can find a matching $N$ of $H$ covering every vertex of $\mathcal{B}_1$ and we complete the proof. □

Let $e \in E$ and let $G_1^e$ and $G_2^e$ be subgraphs of $G^I$ induced by $\overline{\mathcal{B}_e}$ and $\mathcal{B}_e$, respectively. By the definition of intersection graphs, we have the following result.

**Lemma 2.8.** *Let $M = (E, \mathcal{B})$ be a matroid on $E$ and $G^I$ be the intersection graph for bases of $M$. For any $e \in E$, $G_1^e$ is the intersection graphs for bases of $M \backslash e$ and $G_2^e = G^I - V(G_1^e)$.*

It is easy to see that $G_2^e$ is a complete graph induced by the vertices containing $e$ and $|V(G_2^e)| = |\mathcal{B}_e|$. If $r(M) \geq 2$, then the intersection graph $G^I$ has the base graph $G'$ of $M$ as a connected spanning subgraph. By Lemmas 2.7 and 2.8, we have the following lemma immediately.

**Lemma 2.9.** *Let $M = (E, \mathcal{B})$ be a simple matroid and $e \in E$. Let $G^I$ be the intersection graph for bases of $M$. If $r(M) = 2$ and $|E| = n \geq 5$ or $r(M) \geq 3$ and $|E| = n \geq 2r$, then for any four distinct vertices $B_1$, $B_2$, $B_3$ and $B_4$ of $V(G_1^e)$, there exist four distinct vertices $B_1'$, $B_2'$, $B_3'$ and $B_4'$ of $V(G_2^e)$ such that $B_1B_1'$, $B_2B_2'$, $B_3B_3'$ and $B_4B_4'$ are edges of $G^I$.*

**Lemma 2.10.** *Let $M = (E, \mathcal{B})$ be a simple matroid on $E$. If $|E| = n \geq 4$ and $r(M) = 2$, then the intersection graph $G^I$ for bases of $M$ has at least two edge-disjoint Hamilton cycles.*

**Proof:** It is easy to see that $M$ is isomorphic to $U_{2,n}(n \geq 4)$. So we show that the intersection graph $G^I(U_{2,n})$ $(n \geq 4)$ has two edge-disjoint Hamilton cycles. We prove this by induction on $n$. If $n = 4$, then $M$ is isomorphic to $U_{2,4}$. Set $E = \{1, 2, 3, 4\}$. Then $\mathcal{B}(U_{2,4}) = \{(12), (13), (14), (23), (24), (34)\}$. We label the vertices of $G^I$ by the bases of $U_{2,4}$.

Clearly, (12)(13)(23)(24)(34)(14)(12) and (12)(23)(34)(13)(14)(24)(12) are two edge disjoint Hamilton cycles of $G^I$ (see Fig.1).

Suppose that the result holds for $|E| \leq n - 1$. Next we prove that the result holds for $|E| = n \geq 5$. Set $E = \{1, 2, \cdots, n\}$. For $n \in E$, the induction hypothesis assures the existence of two edge-disjoint Hamilton cycles $C_1$ and $C_2$ in $G_1^n$ because $G_1^n$ is isomorphic to $G^I(U_{2,n}\backslash n)$.

By Lemma 2.5 and $|V(G_1^n)| = C_{n-1}^2 \geq C_4^2 \geq 6$, we can choose $e_1 = B_1B_2 \in E(C_1)$ and $e_2 = B_3B_4 \in E(C_2)$ such that $\{e_1, e_2\}$ is a matching of $G^I$. By Lemma 2.9, we can find four distinct vertices $B_1'$, $B_2'$, $B_3'$ and $B_4'$ in $G_2^n$ such that $B_1B_1'$, $B_2B_2'$, $B_3B_3'$ and $B_4B_4'$ are the edges of $G^I$. By Lemma 2.2 and $|V(G_2^n)| \geq 4$, there are two edge-disjoint Hamilton paths $P_1$ and $P_2$ with $\{B_1', B_2'\}$ and $\{B_3', B_4'\}$ as their end vertices because $G_2^n$ is a complete graph $K_m$ with $m = C_n^1 = n \geq 5$. Then $(C_1 - B_1B_2) \cup B_1B_1' \cup P_1 \cup B_2B_2'$ and $(C_2 - B_3B_4) \cup B_3B_3' \cup P_2 \cup B_4B_4'$ are two edge-disjoint Hamilton cycles in $G^I$ (see Fig. 2). Hence we complete the proof. □

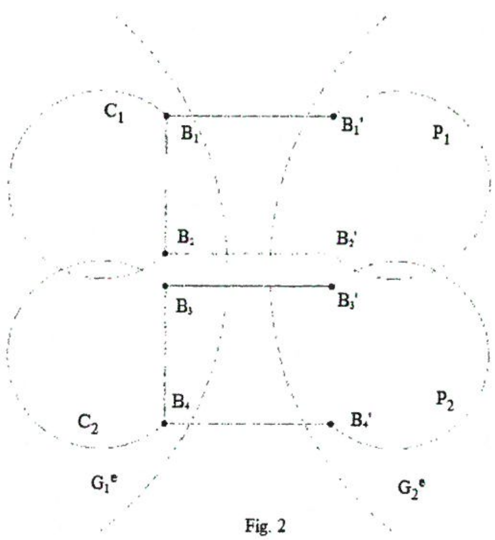


Fig. 2

**Proof of Theorem 2.1:** We prove the theorem by induction on $|E| = n$. Note that $|E| > 3$ by the hypothesis. When $r(M) = 2$, by Lemma 2.10, the theorem holds. When $|E| = 5$ and $3 \leq r(M) \leq 4$, $G^I$ is a complete graph $K_m$. Since $m \geq 5$, $G^I$ has at least two edge-disjoint Hamilton cycles.

Assume that the theorem is true for $|E| \leq n-1$. We show that the theorem holds for $|E| = n \geq 6$. When $n < 2r$, $G^I$ is a complete graph $K_m$ with $m \geq 5$ and thus has two edge-disjoint Hamilton cycles. Next we consider the case $n \geq 2r \geq 6$.

Let $e$ be a given element of $E$. If $M \backslash e$ has coloops, by Lemma 2.6, we have $|V(G_1^e)| \geq 2(n-r-1)+1 \geq 2(r-1)+1 \geq 5$; then $G_1^e$ is a complete graph $K_m$ with order $m \geq 5$. So $G_1^e$ has two edge-disjoint Hamilton cycles $C_1$ and $C_2$. If $M \backslash e$ does not have coloops, then the induction hypothesis assures that $G_1$ also has two edge-disjoint Hamilton cycles $C_1$ and $C_2$.

By Lemma 2.5, we can choose $e_1 = B_1B_2 \in E(C_1)$ and $e_2 = B_3B_4 \in E(C_2)$ such that $\{e_1, e_2\}$ is a matching of $G_1^e$. By Lemma 2.9, there exist four distinct vertices $B_1'$, $B_2'$, $B_3'$ and $B_4'$ of $G_2^e$ such that $B_1B_1'$, $B_2B_2'$, $B_3B_3'$ and $B_4B_4'$ are edges of $G^I$. By Lemma 2.2 and $|V(G_2)| \geq n-r+1 \geq r+1 \geq 4$, that have two edge disjoint Hamilton paths $P_1$ and $P_2$ in $G_2^e$ that have $\{B_1', B_2'\}$ and $\{B_3', B_4'\}$ as their end vertices, respectively. Thus $(C_1 - B_1B_2) \cup B_1B_1' \cup P_1 \cup B_2B_2'$ and $(C_2 - B_3B_4) \cup B_3B_3' \cup P_2 \cup B_4B_4'$ are two edge disjoint Hamilton cycles in $G^I$ (see Fig. 2). Hence we complete the proof. □

**Remark.** When $r(M) = 2$, the intersection graph $G^I(M)$ is the same as the base graph $G'(M)$. By Lemma 2.10, we can conclude that base graph $G'(M)$ of a simple matroid $M$ with rank $r(M) = 2$ has at least two edge-disjoint Hamilton cycles whenever $|E| = n \geq 4$. The example shown in Fig. 1 has only two edge-disjoint Hamilton cycles, which demonstrate that Theorem 2.1 is best possible. For simple matroids $M$ with rank $r(M) \geq 3$, we anticipate that $G'(M)$ has at least two edge-disjoint Hamilton cycles when $|V(G'(M))|$ is relatively large. Note that $M$ needs to be a simple matroid; otherwise, consider a matroid $M$ consisting of three connected components and each component is a 2-circuit. Then we can see that the base graph of $M$ is the cube $C_4 \square K_2$, which does not contain two edge-disjoint Hamiltonian cycles.

Since all the known examples without two edge-disjoint Hamilton cycles have relatively small orders, this prompts us to pose an open problem:

**Open Problem 2.11.** *Let $G'$ be the base graph for a simple matroid with rank $r(M) \geq 3$. Does there exist a constant $N_0$ such that $G'$ contains two or more edge-disjoint Hamilton cycles when $|V(G'(M))| \geq N_0$?*

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