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ORIGINAL PAPER

On Superconnectivity of (4, g)-Cages

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Abstract A (k, g)-*cage* is a graph that has the least number of vertices among all k-regular graphs with girth g. It has been conjectured (Fu et al. in J. Graph Theory, 24:187–191, 1997) that all (k, g)-cages are k-connected for every $k \geq 3$. A k-connected graph G is called *superconnected* if every k-cutset S is the neighborhood of some vertex. Moreover, if $G - S$ has precisely two components, then G is called *tightly superconnected*. In this paper, we prove that every $(4, g)$-cage is tightly superconnected when $g \geq 11$ is odd.

Keywords Cage · Superconnected · Tightly superconnected

1 Introduction

Throughout this paper, only undirected simple graphs are considered. Unless otherwise defined, we follow [1] for terminology and definitions.

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Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $u, v \in V(G)$, $d_G(u, v)$ denotes the length of a shortest path between u and v in G. For vertex sets $T_1, T_2 \subseteq V(G)$, $E(T_1, T_2)$ is the set of the edges with end-vertices in T_1 and T_2, respectively, and $d(T_1, T_2) = d_G(T_1, T_2) = min\{d_G(t_1, t_2) : t_1 \in T_1, t_2 \in T_2\}$ denotes the *distance* between T_1 and T_2. For $S \subset V(G)$, $G - S$ is the subgraph of G obtained by deleting the vertices in S and all the edges incident with them. The set of vertices which are at distance r to S in G is denoted by $N_r(S) = \{v \in V(G) : d_G(v, S) = r\}$, where r is an integer. We write $N(S)$ instead of $N_1(S)$. The length of a shortest cycle in G is called the *girth* of G, denoted by $g(G)$. The *diameter* of G is the maximum distance between any two vertices in G. Let $G[S]$ be the induced subgraph of G for $S \subseteq V(G)$.

A k-regular graph with girth g is called a (k, g)-*graph*. A (k, g)-*cage* is a (k, g)-graph with the least number of vertices for given k and g. We use $f(k, g)$ to denote the number of vertices of a (k, g)-cage. A cutset X of G is called a *non-trivial cutset* if X does not contain the neighborhood $N(u)$ of any vertex $u \notin X$. A k-connected (or k-vertex-connected) graph G is called *superconnected* if for every vertex cutset $S \subseteq V(G)$ with $|S| = k$ is a trivial cutset. The *superconnectivity* of G is denoted by $\kappa_1 = \kappa_1(G) = \min\{|X| : X \text{ is a non-trivial cutset}\}$. Moreover, if $G - S$ has precisely two components, then G is called *tightly superconnected*. The edge-superconnectivity λ_1 is defined similarly.

Cages were introduced by Tutte [14] in 1947, and have been extensively studied. Most of the work carried out so far has focused on the existence problem, whereas very little is known about the structural properties of (k, g)-cages. For more information, reader is referred to the surveys [4,16]. Recently, several researchers have studied the connectivity of cages. Fu et al. [5] proved that all cages are 2-connected, and then subsequently showed that all cubic cages are 3-connected. They then conjectured that (k, g)-cages are k-connected. Daven and Rodger [2], and independently Jiang and Mubayi [6], proved that all (k, g)-cages are 3-connected for $k \geq 3$. Xu et al. [17] proved that every $(4, g)$-cage is 4-connected, and Marcote et al. [12] improved this result in showing that every (k, g)-cage with $k \geq 4$ is 4-connected. Further, Lin et al. [8] have proved that every (k, g)-cage with $k \geq 3$ and odd girth $g \geq 7$ is $\lceil \sqrt{k+1} \rceil$-connected.

For the edge-connectivity of (k, g)-cages, Wang et al. [15] showed that (k, g)-cages are k-edge-connected when g is odd, and subsequently, Lin et al. [9] proved that (k, g)-cages are k-edge-connected when g is even. Recently, Lin et al. [7] and Marcote and Balbuena [10] proved that (k, g)-cages are edge-superconnected.

The objective of this paper is to prove that every $(4, g)$-cage with odd girth is tightly superconnected. Cubic cages have been shown to be tightly superconnected in [11].

2 Main Results

First, we list several known results which will be used in proving our main theorem.

Theorem 1 [3,5] *Let G be a (k, g)-cage with diameter D, where $k \geq 2$ and $g \geq 3$. Then $D \leq g$ and $f(k, g) < f(k, g+1)$.*

Theorem 2 [10] *Every (k, g)-cage with odd girth $g \geq 5$ is edge-superconnected.*

For edge-connectivity, Tang et al. [13] made the following conjecture:

Conjecture 1 [13] *Every* (k, g)*-cage of odd girth* $g \geq 5$ *has* $\lambda_1 = 2k - 2$.

Here we verify the conjecture for $k = 4$.

Lemma 1 *Every* $(4, g)$*-cage of girth* $g \geq 5$ *has* $\lambda_1 = 6$.

Proof Let M be a non-trivial minimum edge cutset of G. From Theorem 2, every $(4, g)$-cage is edge-superconnected and thus $|M| \geq 5$. Suppose C is a component of $G - M$. The degree sum of all the vertices in C should be even, i.e., $4|V(C)| - |M| = \sum_{v \in V(C)} d_C(v) \equiv 0 \pmod 2$. Thus $|M|$ must be even and so $|M| \geq 6$. Moreover, if $uv \in E(G)$, then $E(\{u, v\}, N(\{u, v\}))$ is a nontrivial edge cutset of G, of cardinality 6. As a consequence, $\lambda_1(G) = 6$. □

The following lemma has been proved in [13].

Lemma 2 [13] *Let* G *be a* $(4, g)$*-cage with odd girth* $g \geq 5$. *Assume that there exists a non-trivial cutset* $X \subseteq V(G)$ *such that* $|X| = 4$, *and let* C *be a component of* $G - X$. *Then there exists a vertex* $u \in V(C)$ *such that* $d(u, X) \geq (g - 1)/2$.

We now provide a stronger version of the above lemma.

Lemma 3 *Let* G *be a* $(4, g)$*-cage with odd girth* $g \geq 5$. *Assume that there exists a non-trivial cutset* $X \subseteq V(G)$ *such that* $|X| = 4$, *and let* C *be a component of* $G - X$. *Then* $\max\{d(u, X) : u \in V(C)\} = (g - 1)/2$.

Proof By Lemma 1 we know that $\lambda_1 = 6$, then $G - X$ contains exactly two components C and C'. By Lemma 2, there exists a vertex $v \in V(C')$ such that $d_{C'}(v, X) \geq (g - 1)/2$. Since the diameter of G is at most g, there exists a vertex $u \in V(C)$ such that $(g + 1)/2 \geq d(u, X) \geq (g - 1)/2$. Suppose $d(u, X) = (g + 1)/2$, then $d(v, X) = (g - 1)/2$. Let $N_C(u) = \{u_1, u_2, u_3, u_4\}$, $N_{C'}(v) = \{v_1, v_2, v_3, v_4\}$ and $X = \{x_1, x_2, x_3, x_4\}$. Then $d(u_i, v_j) \geq g - 2$, for all $i, j = 1, 2, 3, 4$.

Claim 1 *For each* $x \in X$, *if* $d(x, N(v)) = (g - 3)/2$, *then there exists a unique* $v' \in N(v)$ *such that* $d(x, v') = (g - 3)/2$.

Otherwise, suppose $d(x, v_1) = d(x, v_2) = (g - 3)/2$, then a cycle of length shorter than g is formed by the two shortest paths from x to v_1 and v_2 together with vv_1 and vv_2.

Claim 2 *There exist* $u_n, u_p \in N(u)$ *and distinct* $v_m, v_q \in N(v)$ *such that* $d(u_n, v_m) \geq g - 1$ *and* $d(u_p, v_q) \geq g - 1$.

Otherwise, assume that there exists at most one vertex $s \in N(u) \cup N(v)$, such that $d(u_i, v_j) = g - 2$, for all $u_i, v_j \neq s$. Then taking into account Claim 1, each vertex in $N(u) - s$ is at distance $(g - 1)/2$ from each vertex in X, and there are at least twelve shortest paths of length $(g - 1)/2$ from $N(u) - s$ to X, which can not have a common vertex in $(N(X) \cap V(C)) - X$ (otherwise, a cycle of length shorter than g appears in G). So $|E(X, C)| \geq 12$ and then there are at most four edges left from X to C', a contradiction to Lemma 1.

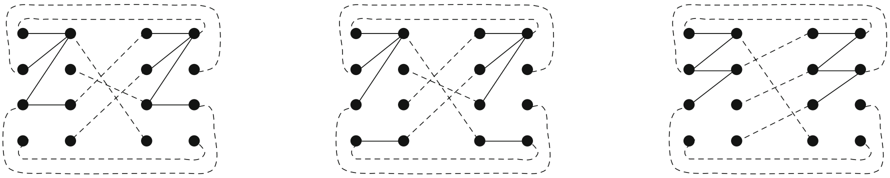

Fig. 1 Graph $G \cup E(f) \cup E(f^*)$

Without loss of generality, by Claim 2, we assume $d(u_1, v_1) \geq g - 1$ and $d(u_2, v_2) \geq g - 1$. Then we can construct a new $(4, g')$-graph as follows: in $G' = G - u - v$, add a vertex y and six edges u_1v_1, u_2v_2, yu_3, yu_4, yv_3 and yv_4. So $|V(G')| < |V(G)|$, and it is clear that $g' \geq g$, a contradiction to Theorem 1. □

Suppose U and W are two vertex subsets of a given graph and $|U| = |W|$. For a 1–1 mapping $f : U \mapsto W$, we define $E(f) = \{uf(u) : u \in U\}$.

Lemma 4 *Let H be a bipartite graph with bipartition (U, W), where $|U| = |W| = 4$, such that $|E(H)| \leq 4$ and $\Delta(H) \leq 3$. Let H^* be a copy of H with bipartition (U^*, W^*) and $G = H \cup H^*$. Then there exist two 1–1 mappings $f : W \mapsto U^*$ and $f^* : W^* \mapsto U$ such that no new 4-cycle is created in graph $G \cup E(f) \cup E(f^*)$.*

Proof Let $U = \{a_1, b_1, c_1, d_1\}$ and $W = \{a_2, b_2, c_2, d_2\}$. It suffices to show that the result holds for $|E(H)| = 4$ and $\Delta(H) \leq 3$. Let f^* be defined by $E(f^*) = \{a_2^*a_1, b_2^*b_1, c_2^*c_1, d_2^*d_1\}$, where $a_i^*, b_i^*, c_i^*d_i^*$ denote the copies of a_i, b_i, c_i, d_i $(i = 1, 2)$. Let us define the other 1–1 mapping f according to the following cases. First, if H can be partitioned into two disconnected bipartite subgraphs $H_1 = (\{a_1, b_1\}, \{a_2, b_2\})$ and $H_2 = (\{c_1, d_1\}, \{c_2, d_2\})$ of cardinality four, then f is defined by $E(f) = \{a_2c_1^*, b_2d_1^*, c_2a_1^*, d_2b_1^*\}$. Second, if H has a vertex of degree 3, say $a_2a_1, a_2b_1a_2c_1 \in E(H)$ (see the two graphs depicted on the left in Fig. 1 in which the pair of 1–1 mappings are indicated by dotted lines) or H contains a path of length 4 (see the graph depicted on the right in Fig. 1), then $E(f) = \{a_2d_1^*, b_2c_1^*, c_2b_1^*, d_2a_1^*\}$. In either case, it is easy to verify that G has no 4-cycles. □

To prove that every $(4, g)$-cage G with odd girth $g \geq 11$ is tightly superconnected, we reason by contradiction and assume that there exists a non-trivial cutset S of order 4 in G. Let G_1 be the smaller component of $G - S$ and $G_2 = G - S - G_1$. Then, from Lemma 3, we see that $\max\{d(u, S) : u \in V(G_i)\} = (g - 1)/2$ $(i = 1, 2)$ and $|V(G_1)| \leq |V(G)|/2 - 2$. We proceed by constructing a $(4, g')$-graph of order less than $|V(G)|$, where $g' \geq g$, which contradicts Theorem 1. To do that, the following consequence is quite useful.

Corollary 1 *(i) Let $N(u) = \{u_1, u_2, u_3, u_4\}$, then*

$$(g - 5)/2 \leq d_{G_1}(N(u_i) - u, S) \leq (g - 3)/2 \text{ for all } i = 1, 2, 3, 4. \quad (1)$$

(ii) Given $s \in S$ such that $d(s, u_{ij}) \leq (g - 3)/2$ for some $u_{ij} \in N(u_i)$, then $d(s, u') \geq (g - 1)/2$ for all $u' \in N(u_i) - u_{ij}$.

(iii) $|N_{(g-5)/2}(s) \cap N_2(u)| \leq 1$ *for all* $s \in S$ *and* $|N_{(g-5)/2}(x) \cap N_2(u)| \leq 1$ *for all* $x \in N(S)$.

Proof If (i) does not hold, then the vertex u_i is at distance $(g+1)/2$ to S, which is impossible by Lemma 3. If (ii) or (iii) is not true, then a cycle of length $g-1$ can be created. □

Lemma 5 *If* $|V(G_1) \cap N(s_i)| = 2$ *and* $|V(G_2) \cap N(s_i)| = 2$ *for all* $s_i \in S$*, then* G *is not a* $(4, g)$*-cage.*

Proof Let $N(u) = \{u_1, u_2, u_3, u_4\}$ and $W_i = N(u_i) - u = \{u_{i1}, u_{i2}, u_{i3}\}$ for $i = 1, 2, 3, 4$.

Claim 1 *For each* W_i, $i = 1, 2, 3, 4$*, if there exists at most one vertex* $x_j \in W_i$ *such that* $d(x_j, S) = (g-5)/2$*, then* G *is not a* $(4, g)$*-cage.*

Any two vertices from W_i are not at distance $(g-3)/2$ to the same vertex in S. Otherwise, a cycle of length $g-1$ appears. Similarly, it is impossible to have $d(u_i, s) = d(u_j, s) \leq (g-3)/2$ for any two distinct vertices u_i, u_j and a vertex $s \in S$. And there is a vertex in W_i which is at distance $(g-5)/2$ or $(g-3)/2$ to S. Otherwise, the vertex u_i is at distance $(g+1)/2$ to S which is impossible by Lemma 3. Without loss of generality, assume $u_{i1} \in W_i$ to be a vertex that satisfies $d(u_{i1}, S) \in \{(g-5)/2, (g-3)/2\}$. In the rest of this paper, *connecting two vertices* means joining the two vertices by a new edge and *connecting a vertex x to a set R* means joining x to every vertex in R.

Let $W = \{z_1, z_2, z_3, z_4\}$. We construct a bipartite graph $H = (W, S)$, where $|W| = |S| = 4$ and $z_i s_j \in E(H)$ if and only if $d_{G_1}(s_j, W_i - u_{i1}) \leq (g-3)/2$. It is clear that there are at most eight paths in G of length at most $(g-3)/2$ from $\cup_{i=1}^4 W_i$ to S; otherwise, since $|V(G_1) \cap N(s_i)| = 2$ $(i = 1, 2, 3, 4)$, containing more than eight paths implies that a cycle of length shorter than g appears. This implies that there are at most four paths of length at most $(g-3)/2$ from $\cup_{i=1}^4 (W_i - u_{i1})$ to S. Hence we have that $|E(H)| \leq 4$ and furthermore, we see that $\triangle(H) \leq 3$, because these four paths can not start from the same $W_i - u_{i1}$, otherwise, by the Pigeonhole Principle, it would imply that u_{i1} and another vertex from W_i have distance $(g-3)/2$ to the same vertex in S, which is impossible.

Now for the bipartite graph H, we showed that $\triangle(H) \leq 3$ and $|E(H)| \leq 4$. Let H^* be a copy of H. By Lemma 4, there are two 1–1 mappings $f : S \mapsto W^*$ and $f^* : S^* \mapsto W$ such that no new 4-cycles are created in $H \cup H^* \cup E(f) \cup E(f^*)$.

Considering the subgraph $N = G[(V(G_1) - u - N(u)) \cup S]$, each edge in $H \cup H^*$ implies a path of length $(g-3)/2$ in graph N, and the existence of mappings f and f^* implies that there is a way to connect two copies of N such that there exists no cycles of length $(1 + 1 + (g-3)/2 + (g-3)/2) = g - 1$, which is corresponding to a 4 cycle in $H \cup H^* \cup E(f) \cup E(f^*)$.

Let N^* be a copy of N. For every $x \in V(N)$, let x^* denote its copy in N^*. Now we construct a 4-regular graph G' (see Fig. 2) with girth at least g by using N and N^*:

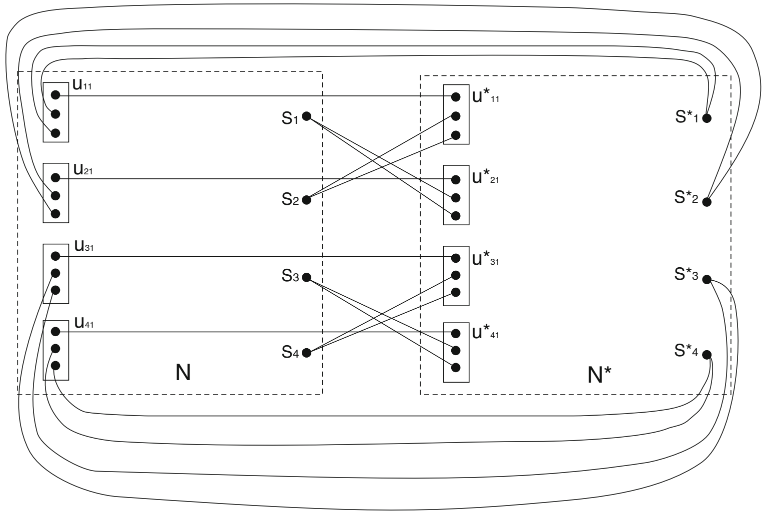

Fig. 2 Illustration of the construction in Claim 1, where $f^*(s_i^*) = z_i (i = 1, 2, 3, 4)$ and $f(s_1) = z_2^*$, $f(s_2) = z_1^*$, $f(s_3) = z_4^*$, $f(s_4) = z_3^*$

(a) connect u_{i1} and u_{i1}^* for $i = 1, 2, 3, 4$;
(b) s_i is connected with u_{j2}^* and u_{j3}^* if and only if $f(s_i) = W_j'^*$ for $i, j = 1, 2, 3, 4$;
(c) s_i^* is connected with u_{j2} and u_{j3} if and only if $f^*(s_i^*) = W_j'$ for $i, j = 1, 2, 3, 4$.

Consider the girth of G'. Any new cycle C introduced in the construction has to use at least two new edges added in the processes (a), (b) and (c). If C goes through two edges in (a), then C has length at least $2(g-4)+2 > g$ since $g \geq 11$. If C contains two edges in (b) and (c), then the length of C is at least $(g-1)/2+(g-3)/2+2 = g$, because $H \cup H^* \cup E(f) \cup E(f^*)$ creates no new 4-cycles. If C goes through one edge in (a) and one edge in (b) or (c), then C has the length at least $(g-4)+2+(g-3)/2 > g$ since $g \geq 11$. It is obvious that if the cycle C goes through more than two new edges, its length is at least g. Hence G' is 4-regular and has girth at least g, but $|V(G')| = |V(N^*)| + |V(N)| = 2|V(G_1)| - 2 < |V(G)|$, a contradiction to the fact that G is a cage. So Claim 1 is proved.

We continue the proof by considering two cases according to the neighbors of u.

Case 1 All the neighbors of u are at distance $(g-3)/2$ to S.

This is a special case of Claim 1.

Case 2 There are at most three neighbors of u at distance $(g-3)/2$ to S.

Hence there exists a vertex $v \in N(u)$ such that $d(v, S) = d(u, S) = (g-1)/2$. Let $N(u) = \{u_1, u_2, u_3, v\}$, $N(v) = \{v_1, v_2, v_3, u\}$, $W_i = N(u_i) - u = \{u_{i1}, u_{i2}, u_{i3}\}$ and $T_i = N(v_i) - v = \{v_{i1}, v_{i2}, v_{i3}\}$, $i = 1, 2, 3$. If there is at most one vertex $x \in W_i$ or $y \in T_i$ such that $d(x, S) = d(y, S) = (g-5)/2$ for $i = 1, 2, 3$, then by Claim 1, G is not a (4, g)-cage.

Assume that there exist two sets W_i and T_j, say W_3 and T_3, such that $|N_{(g-5)/2}(S) \cap T_3| \geq 2$ and $|N_{(g-5)/2}(S) \cap W_3| \geq 2$.

Now we consider W_i and T_i for $i = 1, 2$. Let us examine the distance from W_i and T_i to S. If, say in W_1, there are no vertices with distance less than $(g-1)/2$ to S, then $d(u_1, S) = (g+1)/2$, contradicting to Lemma 3. If the shortest path from W_1 to S is of length $(g-3)/2$, then there exist at least two paths from W_1 to S of length $(g-3)/2$; otherwise, by applying Claim 1 on vertex u_1, which is at distance $(g-1)/2$ to S, we see that G is not a $(4, g)$-cage. Another possibility is that there exists a path of length $(g-5)/2$ from W_1 to S. So we may assume that, for each W_i and T_i $(i = 1, 2)$, either there exists a path of length $(g-5)/2$ or there are two paths of length $(g-3)/2$ to S. Let $F = (\cup_{i=1}^{4} N(s_i) \cap V(G_1))$ and so $|F| \leq 8$, and let $\mathcal{P}$ be the set of paths from $\cup_{i=1}^{3}(W_i \cup T_i)$ to S, of length $(g-5)/2$ or $(g-3)/2$. Consider any vertex $x \in F$. Note that if some path in $\mathcal{P}$ of length $(g-5)/2$ goes through x, then there are no other paths from $\mathcal{P}$ through this vertex, i.e., this path is unique (otherwise, some cycle of length less than g appears). The girth condition also assures that at most two paths in $\mathcal{P}$ of length $(g-3)/2$ can go through vertex x, one of them starting at $\cup_{i=1}^{3} W_i$ and the other one in $\cup_{i=1}^{3} T_i$.

Suppose, in $\mathcal{P}$, that there are m_1 paths of length $(g-5)/2$ and m_2 paths of length $(g-3)/2$. Since $|F| \leq 8$, for each vertex in F, there are at most two paths of length $(g-3)/2$ in $\mathcal{P}$ going through it, or there is only one path of length $(g-5)/2$ in $\mathcal{P}$ going through it, therefore we have $2m_1 + m_2 \leq 16$.

And we know that $|N_{(g-5)/2}(S) \cap T_3| = |N_{(g-5)/2}(S) \cap W_3| = 2$ by the assumption, which implies $m_1 \geq 4$. As well, there are either one path of length $(g-5)/2$ or two paths of length $(g-3)/2$ to S from each W_i and T_i $(i = 1, 2)$. Therefore $2m_1 + m_2 = 16$ and there are no other paths of length shorter than $(g-1)/2$ from $W_i \cup T_i$ $(i = 1, 2, 3)$ to S. Hence, there are exactly two paths of length less than $(g-1)/2$ from W_3 to S. Without loss of generality, assume $d(u_{31}, s_1) = (g-5)/2$ and $d(u_{32}, s_2) = (g-5)/2$. We also know $d(u_{33}, S) \geq (g-1)/2$, $d(u_{31}, S - s_1) \geq (g-1)/2$ and $d(u_{32}, S - s_2) \geq (g-1)/2$.

Let $L_{3i} = N(u_{3i}) - u_3 = \{u_{3i1}, u_{3i2}, u_{3i3}\}$ $(i = 1, 2, 3)$, and $d(u_{311}, s_1) = (g-7)/2$ and $d(u_{321}, s_2) = (g-7)/2$. Apart from these two paths of length $(g-7)/2$ there are no other paths of length less than $(g-1)/2$ joining $\{L_{31}, L_{32}, L_{33}, u_1, u_2, v\}$ to $\{s_1, s_2\}$ and there are at most four paths of length $(g-3)/2$ from $\{L_{31}, L_{32}, L_{33}, u_1, u_2, v\}$ to $\{s_3, s_4\}$ due to the girth condition (for each s_j, one path from $\cup_{i=1}^{3} L_{3i}$ to s_j and another path from $\{u_1, u_2, v\}$ to s_j). Suppose $d(u_{331}, S) = d(L_{33}, S)$ and $d(u_1, S) = d(N(u) - u_3, S)$. Let $N = G[(G_1 - u_3 - N(u_3)) \cup S]$ and N^* be a copy of N. Now we construct a new 4-regular graph $G' = N \cup N^* \cup M$ (see Fig. 3), where M is the set of edges defined as:

(a) connect u_{3i1} and u_{3i1}^* $(i = 1, 2, 3)$, u_1 and u_1^*;
(b) let $b_1 = L_{31} - u_{311}, b_2 = L_{32} - u_{321}, b_3 = L_{33} - u_{331}, b_4 = \{u_2, v\}$, $B = \{b_1', b_2', b_3', b_4'\}$, and let $\Gamma = (B, S)$ be a bipartite graph defined as follows: $b_i' s_j \in E(\Gamma)$ if and only if $d(b_i, s_j) \leq (g-3)/2$. It is easy to see that Γ satisfy the conditions of Lemma 4, and thus there is a way to connect N and N^* without creating small cycles.
(c) then other edges are added in a similar fashion as in Claim 1.

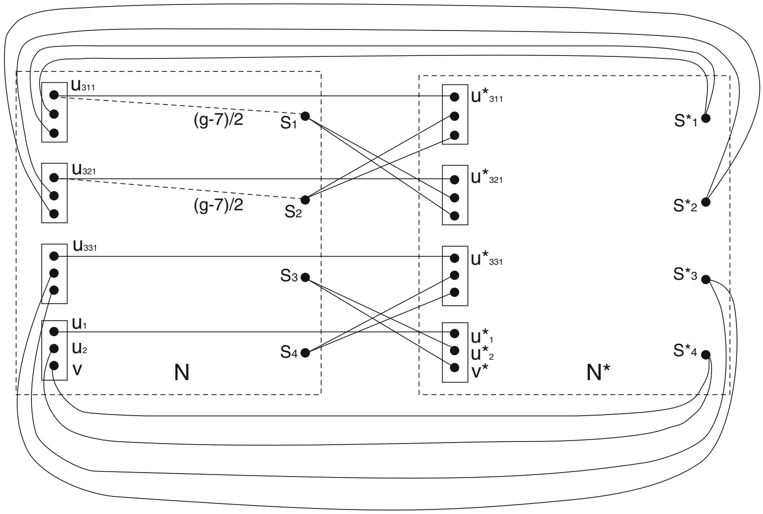

Fig. 3 Illustration of the construction in Case 2

The length of any new cycle containing $u^*_{3i1}u_{3i1}$ or $u_1u^*_1$ is at least $(g-7)/2+2+(g-4)\geq g$ $(i=1,2,3)$ since $g\geq 11$. Other new cycles are of length at least g as shown in Claim 1. Moreover $|V(G')|=2|V(G_1)|-2<|V(G)|$. Hence G' is a 4-regular graph and its girth is at least g, a contradiction to the fact that G is a $(4,g)$-cage. □

Lemma 6 *If* $|V(G_1)\cap N(s_1)|=|V(G_1)\cap N(s_2)|=3$, $|V(G_2)\cap N(s_1)|=|V(G_2)\cap N(s_1)|=1$ *and* $|V(G_1)\cap N(s_3)|=|V(G_1)\cap N(s_4)|=|V(G_2)\cap N(s_3)|=|V(G_2)\cap N(s_4)|=2$, *where* $s_i\in S$, *then* G *is not a* $(4,g)$*-cage.*

Proof Let $N(u)=\{u_1,u_2,u_3,u_4\}$ and $W_i=N(u_i)-u=\{u_{i1},u_{i2},u_{i3}\}$, $i=1,2,3,4$.

Claim 1 *If there is at most one path of length* $(g-5)/2$ *from each* W_i *to* $S(i=1,2,3,4)$, *then* G *is not a* $(4,g)$*-cage.*

Without loss of generality, we may assume $d(\{W_1,W_2\},\{s_3,s_4\})\geq(g-3)/2$ and $d(\{W_3,W_4\},\{s_1,s_2\})\geq(g-3)/2$. Then we have

$$d(\{u_1,u_2\},\{s_3,s_4\})\geq\frac{g-1}{2},\quad d(\{u_3,u_4\},\{s_1,s_2\})\geq\frac{g-1}{2}. \tag{2}$$

Suppose that $d(u_{31},s_3)\leq(g-3)/2$ and $d(u_{41},s_4)\leq(g-3)/2$, then by Corollary 1 we have

$$d(W_3-u_{31},s_3)\geq\frac{g-1}{2},\quad d(W_4-u_{41},s_4)\geq\frac{g-1}{2}. \tag{3}$$

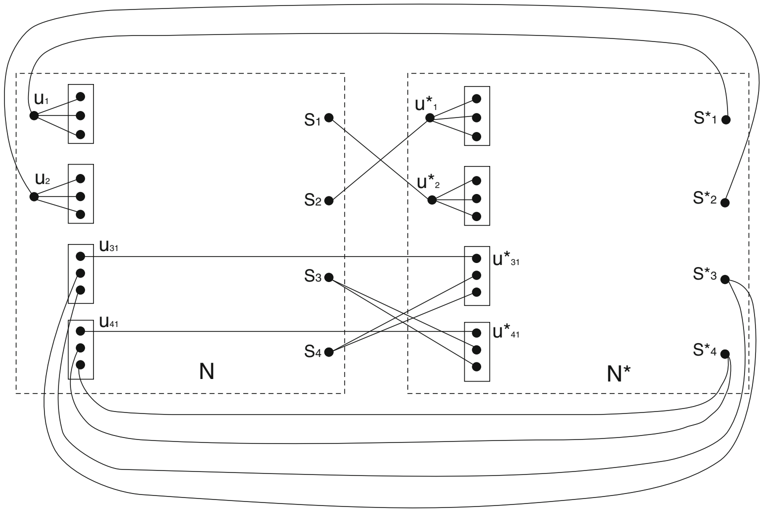

Fig. 4 Illustration of the construction in Claim 1

Moreover, since there is at most one path of length $(g-5)/2$ from each W_i to each vertex in S, we also have

$$d(W_3 - u_{31}, s_4) \geq \frac{g-3}{2}, \quad d(W_4 - u_{41}, s_3) \geq \frac{g-3}{2}. \tag{4}$$

We consider a subgraph N of G_1 induced by $(V(G_1) - \{u, u_3, u_4\}) \cup S$ and let N^* be a copy of N. For every $x \in V(N)$, let x^* denote its copy in N^*. Now we construct a 4-regular graph G' by adding the following edges between N and N^* (see Fig. 4):

(a) connect s_1 and u_2^*, s_2 and u_1^*, s_1^* and u_1, s_2^* and u_2;
(b) connect u_{i1} and u_{i1}^*, $i = 3, 4$;
(c) connect s_3 and the two vertices of $W_4^* - u_{41}^*$;
(d) connect s_4 and the two vertices of $W_3^* - u_{31}^*$;
(e) connect s_3^* and the two vertices of $W_3 - u_{31}$, s_4^* and the two vertices of $W_4 - u_{41}$.

Taking into account (2), (3) and (4), it can be verified that the cycles in the new graph are of length at least g. For instance, if the new edges $s_1^* u_1$ and $u_{31}^* u_{31}$ (or $s_1 u_2^*$ and $u_{31} u_{31}^*$) lie on the same cycle, then this cycle has length at least $(g-5)/2+(g-3)+2 = g$; if the new edges $s_3^* u_{33}$ and $u_2^* s_1$ lie on the same cycle, then this cycle has length $d(u_{33}, s_1) + d(u_2, s_3) \geq (g-3)/2 + (g-1)/2 + 2 = g$ because of (2); or if the new edges $s_3^* u_{33}$ and $u_{42}^* s_3$ lie on the same cycle, then this cycle has length $d(u_{33}, s_3) + d(u_{42}, s_3) \geq (g-1)/2 + (g-3)/2 + 2 = g$ because of (3) and (4). Furthermore, $|V(G')| = |N^*| + |N| = 2|V(G_1)| + 2 \leq |V(G)| - 2$, a contradiction.

Claim 2 *If there is at most one vertex in each W_i at distance $(g-5)/2$ to S, where $i = 1, 2, 3, 4$, then G is not a $(4, g)$-cage.*

Based on Claim 1, we can assume that there is a vertex, say u_{11}, such that $|N_{(g-5)/2}(u_{11}) \cap S| \geq 2$. Suppose $d(u_{11}, s_1) = d(u_{11}, s_2) = (g-5)/2$. Then by Corollary 1

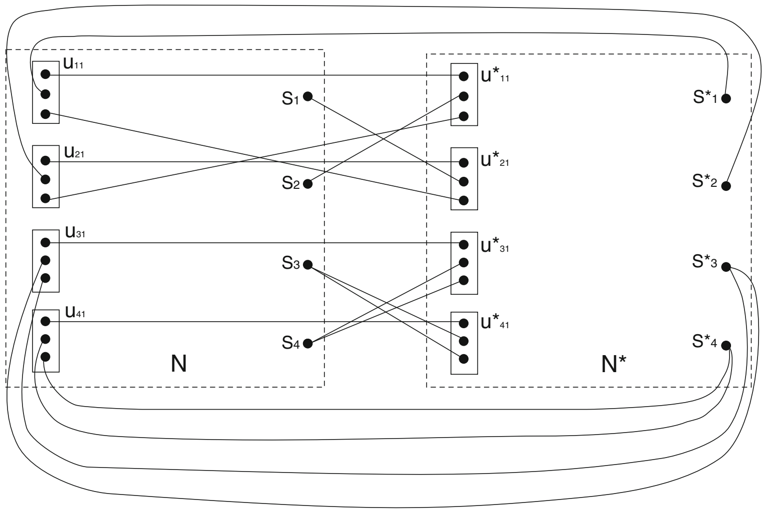

Fig. 5 Illustration of the construction in Claim 2

$$d(W_1 - u_{11}, \{s_1, s_2\}) \geq \frac{g-1}{2}, \quad d(W_1 - u_{11}, \{s_3, s_4\}) \geq \frac{g-3}{2}. \tag{5}$$

Since there are at most four paths of length $(g-5)/2$ from $\cup_{i=1}^4 W_i$ to S, by Pigeonhole Principle there exists a set, say W_2, satisfying

$$d(W_2, S) \geq (g-3)/2. \tag{6}$$

That is, $d_{G_1}(u_2, S) = (g-1)/2$. Therefore, applying Claim 1 to u_2 we may assume that in W_2, there is a vertex, say u_{21}, such that $|N_{(g-3)/2}(u_{21}) \cap S| \geq 2$. Moreover, by Corollary 1 we see that $d(W_3, S) = d(u_{31}, S) \leq (g-3)/2$ and $d(W_4, S) = d(u_{41}, S) \leq (g-3)/2$. Hence (4) is again valid and further, there are four paths of length $(g-5)/2$ or $(g-3)/2$ from W_1, W_3 and W_4 to S, as well as other two paths of length $(g-3)/2$ from u_{21} to S. By the hypothesis on degree distributions of vertices of S, the graph G can only contain in total ten paths of length at most $(g-3)/2$ from $\cup_{i=1}^4 W_i$ to S. Therefore, there are at most four paths of length $(g-3)/2$ from $\cup_{i=1}^4 W_i$ to S left and then there are no paths of length $(g-5)/2$ from $\cup_{i=1}^4 W_i$ to S.

Let N be the subgraph induced by $(V(G_1) - u - N(u)) \cup S$ and N^* be a copy of N. For every $x \in V(N)$, let x^* denote its copy in N^*. Now we construct a 4-regular graph G' by adding the following edges between N and N^* (see Fig. 5):

(a) connect u_{i1} and u_{i1}^*, $i = 1, 2, 3, 4$;
(b) connect s_1 and u_{22}^*, s_2 and u_{12}^*, s_1^* and u_{12}, s_2^* and u_{22};
(c) connect s_3 and the two vertices of $W_4^* - u_{41}^*$;
(d) connect s_4 and the two vertices of $W_3^* - u_{31}^*$;
(e) connect s_3^* and the two vertices of $W_3 - u_{31}$, s_4^* and the two vertices of $W_4 - u_{41}$;
(f) connect all the remaining u_{ij} and u_{ij}^* of degree 3 in $N \cup N^*$.

Taking into account (4), (5) and (6), it can be verified that the cycles in the new graph are of length at least g. Then G' is a $(4, g')$-graph, where $g' \geq g$, but $|V(G')| < |V(G)|$, a contradiction.

In what follows, we consider three cases based on the distance of the neighbors of u to S.

Case 1 All the neighbors of u are at distance $(g-3)/2$ to S.

It follows from Claim 2 that G is not a $(4, g)$-cage.

Case 2 There are exactly three neighbors of u at distance $(g-3)/2$ to S.

Let $N(u) = \{u_1, u_2, u_3, v\}$, $N(v) = \{v_1, v_2, v_3, u\}$, $W_i = N(u_i) - u = \{u_{i1}, u_{i2}, u_{i3}\}$ and $T_i = N(v_i) - v = \{v_{i1}, v_{i2}, v_{i3}\}$, for $i = 1, 2, 3, 4$. Let $d(u, S) = d(v, S) = (g-1)/2$. If there is at least one neighbor of v distinct from u at distance $(g-1)/2$ from S, then we can replace v by u and discuss it as in Case 3 later. So assume $d(v_i, S) = d(u_i, S) = (g-3)/2$ and $d(u_{i1}, S) = d(v_{i1}, S) = (g-5)/2$, for $i = 1, 2, 3$. If $|N_{(g-5)/2}(S) \cap W_i| \leq 1$ or $|N_{(g-5)/2}(S) \cap T_i| \leq 1$ for all $i = 1, 2, 3$, then by Claim 2, G is not a $(4, g)$-cage. Therefore we may assume that $|N_{(g-5)/2}(S) \cap W_3| = |N_{(g-5)/2}(S) \cap T_3| = 2$. This implies that $|N_{(g-5)/2}(S) \cap (\cup_{i=1}^3 W_i \cup T_i)| \geq 8$, since there are at most eight paths of length $(g-5)/2$ from $\bigcup_{i=1}^3 (W_i \cup T_i)$ to S. Hence we have $|N_{(g-5)/2}(S) \cap (\cup_{i=1}^3 W_i \cup T_i)| = 8$.

Claim 3 *For every $X \in \{W_1, W_2, T_1, T_2\}$, there is at least one path of length $(g-3)/2$ from X to S.*

By symmetry, we need only to show that there is at least one path of length $(g-3)/2$ from W_1 to S. Suppose the claim is false, then there is exactly one path of length less than $(g-1)/2$ from W_1 to S, which is the path from u_{11} to S of length $(g-5)/2$. If $d(W_1, \{s_3, s_4\}) = (g-5)/2$, then we regard u_1 as u, and let $N = (G_1 - u_1 - u_{12} - u_{13}) \cup S$ and N^* be a copy of N. Using N and N^*, we construct a new $(4, g')$-graph as in Claim 1 and the new graph is of smaller order and $g' \geq g$, a contradiction.

So we assume $d(W_1, \{s_1, s_2\}) = d(u_{11}, s_1) = (g-5)/2$. Let $r = u_1, r_1 = u_{11}, r_2 = u_{12}, r_3 = u_{13}, r_4 = u$ (i.e., $N(r) = \{r_1, r_2, r_3, r_4\}$) and $R_i = N(r_i) - r = \{r_{i1}, r_{i2}, r_{i3}\}, i = 1, 2, 3, 4$. Then $d(r_2, S) = d(r_3, S) = d(r_4, S) = (g-1)/2$, and we may assume $d(r_{11}, S) = (g-7)/2$. By Claim 2, without loss of generality, assume that there is a vertex in $N(r_i)$, say r_{i1}, such that $|N_{(g-3)/2}(r_{i1}) \cap (S - s_1)| \geq 2$ for $i = 2, 3, 4$. So there is at most one path of length $(g-3)/2$ from $\cup_{i=1}^4 (R_i - r_{i1})$ to $S - s_1$ and no path of length $(g-5)/2$ since $d_{G_1}(s_2) + d_{G_1}(s_3) + d_{G_1}(s_4) = 7$. Thus we can delete $N(r) \cup r$ from G_1 and use a similar construction as in Claim 2 to get a contradiction. So we prove Claim 3.

Since $|N_{(g-5)/2}(S) \cap (\cup_{i=1}^3 W_i \cup T_i)| = 8$, there are at most four paths of length $(g-3)/2$ from $\bigcup_{i=1}^3 (W_i \cup T_i)$ to S. Furthermore, there are exactly two paths of length $(g-5)/2$ and no path of length $(g-3)/2$ from W_3 to S by Claim 3. Denote u by u_{34}. Let $U_i = N(u_{3i}) - u_3 = \{u_{3i1}, u_{3i2}, u_{3i3}\}$ for $i = 1, 2, 3, 4$. Without loss of generality, assume $d(u_{311}, S) = d(u_{321}, S) = (g-7)/2$. If $d(\{u_{311}, u_{321}\}, \{s_1, s_2\}) \geq (g-3)/2$, then using the similar construction as in Claim 1, we get a contradiction. Thus we assume $d(\{u_{311}, u_{321}\}, \{s_1, s_2\}) = (g-7)/2$. Note that there are at least one path of length $(g-3)/2$ from U_4 to S, i.e., $d(u_1, S) = (g-3)/2$. So we have at most four paths of length $(g-3)/2$ from $(U_1 - u_{311}) \cup (U_2 - u_{321}) \cup U_3 \cup (U_4 - u_1)$ to S. Next we use a similar construction as in Claim 2 to yield a contradiction.

Case 3 There are at most two neighbors of u at distance $(g-3)/2$ to S.

Let $N(u) = \{u_1, u_2, u_3 = r, u_4 = v\}$, and assume $d(u, S) = d(v, S) = d(r, S) = (g-1)/2$. Let $W_i = N(u_i) - u = \{u_{i1}, u_{i2}, u_{i3}\}$, for $i = 1, 2$, $W_3 = N(r) - u = \{r_1, r_2, r_3\}$, $W_4 = N(v) - u\{v_1, v_2, v_3\}$, $T_j = N(v_j) - v = \{v_{j1}, v_{j2}, v_{j3}\}$ and $L_j = N(r_j) - r = \{r_{j1}, r_{j2}, r_{j3}\}$ for $j = 1, 2, 3$. Note that $d(W_i, S) = (g-3)/2$ for $i = 3, 4$.

By Corollary 1, there are at most ten paths of length $(g-5)/2$ from $(\cup_{i=1}^{2} W_i) \cup (\cup_{j=1}^{3} T_j) \cup (\cup_{j=1}^{3} L_j)$ to S, since $\sum d_{G_1}(s_i) = 10$. Applying Claim 2 to u, v and r, we may assume that $|N_{(g-5)/2}(S) \cap T_2| \geq 2$, $|N_{(g-5)/2}(S) \cap L_2| \geq 2$ and $|N_{(g-5)/2}(S) \cap W_2| \geq 2$. Furthermore, we may assume either $|N_{(g-5)/2}(S) \cap (W_1 \cup W_2 \cup L_1 \cup L_2 \cup L_3)| \leq 6$ or $|N_{(g-5)/2}(S) \cap (W_1 \cup W_2 \cup T_1 \cup T_2 \cup T_3)| \leq 6$; otherwise, if $|N_{(g-5)/2}(S) \cap (W_1 \cup W_2)| \geq 4$, then we can regard v or r as u instead. Without loss of generality, assume there are at most six paths of length $(g-5)/2$ from $W_1 \cup W_2 \cup T_1 \cup T_2 \cup T_3$ to S. Moreover, we may assume that there are at most three paths of length $(g-5)/2$ from the set $W_1 \cup W_2$ to S and there are also at most three paths of length $(g-5)/2$ from the set $T_1 \cup T_2 \cup T_3$ to S. If not, say there are four paths of length $(g-5)/2$ from $W_1 \cup W_2$ to S, then we can replace $\{u, v\}$ by $\{v_1, v\}$ to have the desired property. Suppose $\{v_1, v\}$ does not have the property we want, then we see $|N_{(g-5)/2}(S) \cap N_2(v_1)| = 4$, $|N_{(g-5)/2}(S) \cap N_2(u)| = 4$ and $|N_{(g-5)/2}(S) \cap N_2(v)| = 2$. Moreover, $|N(S) \cap V(G_1)| = 10$, then we have $d(v_3, S) \geq (g+1)/2$, which yields a contradiction to Lemma 3.

Now we may assume $|N_{(g-5)/2}(S) \cap T_2| = |N_{(g-5)/2}(S) \cap W_2| = 2$ and $|N_{(g-5)/2}(s_j) \cap (W_1 \cup T_1 \cup T_3)| \leq 1$ for $j = 3, 4$. Next we consider two subcases.

Subcase 3.1. There are at most five paths of length $(g-5)/2$ from $\bigcup_{i=1}^{3} (W_i \cup T_i)$ to S.

Suppose $d(W_1 \cup W_3), \{s_3, s_4\}) \geq (g-3)/2$, $d(T_1 \cup T_3, \{s_1, s_2\}) \geq (g-3)/2$, $d(W_1 \cup W_3, \{s_1, s_2\}) = d(W_1, s_1) = d(u_{11}, s_1)$ and $d(T_1 \cup T_3, \{s_3, s_4\}) = d(T_1, s_4) = d(v_{11}, s_4)$. We choose v_{31} such that there is at most one path of length $(g-3)/2$ between $(T_1 - v_{11}) \cup (T_3 - v_{31})$ and $\{s_3, s_4\}$. Moreover, let $d((T_1 - v_{11}) \cup (T_3 - v_{31}), \{s_3, s_4\}) = d(T_1 - v_{11}, s_3)$. Let N be the subgraph of G_1 induced by $(V(G_1) - \{v_1, v_3, u, v\}) \cup S$ and N^* be a copy of N. Now we construct a 4-regular graph G' by adding the following edges between N and N^*:

(a) connect u_2 and u_2^*, v_2 and v_2^*, v_{11} and v_{11}^*, v_{31} and v_{31}^*;
(b) connect s_1 and u_1^*, s_2 and u_3^*, s_1^* and u_3, s_2^* and u_1;
(c) connect s_3 and $T_1^* - v_{11}^*$, s_4 and $T_3^* - v_{31}^*$, s_3^* and $T_3 - v_{31}$, s_4^* and $T_1 - v_{11}$.

The cycle containing edges of type (a) has length at least $(g-5)/2 + 2 + (g-5) \geq g$ or $2 + 2(g-4) \geq g$; the cycle containing the edges in (b) and (c) is at least $(g-3)/2 + (g-1)/2 + 2 \geq g$ or $(g-4) + 4 = g$ or $(g-5) + 2 + 3 \geq g$ or $(g-4) + 3 + 3 \geq g + 2$. Therefore the girth of G' is g.

Clearly, $|V(N) \cup V(N^*)| \leq 2|V(G_1)| < |V(G)|$. So G' is a $(4, g')$-graph of smaller order with $g' \geq g$, a contradiction.

Subcase 3.2. There are six paths of length $(g-5)/2$ from $\cup_{i=1}^{3}(W_i \cup T_i)$ to S.

Without loss of generality, assume $d(W_1, S) = d(u_{11}, S) = (g-5)/2$ and $d(T_1, S) = d(v_{11}, S) = (g-5)/2$. Then $d(u_3, S) = d(v_3, S) = (g-1)/2$. Note that there is at least one path of length $(g-3)/2$ from W_1 to S, and at least one path of length $(g-3)/2$ from T_1 to S. By Claim 1, we can see that there is a vertex, say u_{31}, in W_3 such that there exist two paths of length $(g-3)/2$ from u_{31} to S.

Claim 4 *There are at least three paths of length* $(g-3)/2$ *from* X *to* S*, where* $X \in \{W_3, T_3\}$.

Note that we need only to show that there are at least three paths of length $(g-3)/2$ from W_3 to S. Suppose the claim is false, then there are exactly two paths of length $(g-3)/2$ from W_3 to S. Denote $u = u_{34}$. By Claim 2, we may assume $d(u_{311}, S) = d(u_{312}, S) = (g-5)/2$. Since $d(u_{3i}, S) = (g-1)/2$ for $i = 2, 3$, there is a vertex, say u_{3i1}, in $N(u_{3i})$ such that there are at least two paths of length $(g-3)/2$ from u_{3i1} to S for $i = 2, 3$. There are also three paths of length $(g-3)/2$ from $N(u_{34}) - u_3$ to S. Now we have at most one path of length $(g-3)/2$ from $N_2(u_3) - \{u_{311}, u_{312}, u_{321}, u_{331}, u_{341}, u_{342}\}$ to S. Let $d(\{u_{313}, u_{343}\}, \{s_1, s_2\}) = d(u_{313}, s_1)$ and $d(\{u_{322}, u_{323}, u_{332}, u_{333}\}, \{s_3, s_4\}) = d(u_{322}, s_3)$. Let N be the subgraph of G_1 induced by $(V(G_1) - u_3 - N(u_3))$, and N^* be a copy of N. Now we construct a new graph $G' = N \cup N^* \cup M$, where M is a set of edges connecting between

(a) u_{31i} and u^*_{31i}, u_{34i} and u^*_{34i}, for $i = 1, 2$;
(b) s_1 and u^*_{313}, s_2 and u^*_{343}, s^*_1 and u_{343}, s^*_2 and u_{313};
(c) u_{321} and u^*_{321}, u_{331} and u^*_{331};
(d) s_3 and $\{u^*_{322}, u^*_{323}\}$, s_4 and $\{u^*_{332}, u^*_{333}\}$, s^*_3 and $\{u_{332}, u_{333}\}$, s^*_4 and $\{u_{322}, u_{323}\}$.

With a similar verification as in Claim 2, we conclude that G' is a $(4, g')$-graph, where $g' \geq g$, and $|V(G')| = |V(N)| + |V(N^*)| \leq 2|V(G)| - 6$, a contradiction.

Claim 4 implies that $|N(S) \cap V(G_1)| = 10$; using a similar technique as in the proof of Case 2 we conclude that there are exactly two paths of $(g-5)/2$ and no path of length $(g-3)/2$ from w_2 to S by Claim 4. Then, as in the proof of Case 2, we can construct a $(4, g')$-graph, where $g' \geq g$, such that $|G'| < |G|$.

We complete the proof of this lemma. □

Theorem 3 *Every* $(4, g)$*-cage with odd girth* $g \geq 11$ *is superconnected.*

Proof Suppose G is not superconnected, then we choose a non-trivial cutset S of order 4 such that S minimizes the order of the smaller component of $G - S$ among all non-trivial cutsets. Since $4|V(G_1)| - E(S, G_1) = \sum_{v \in V(G_1)} d_{G_1}(v) \equiv 0 \pmod 2$, we have $E(S, G_1) \equiv 0 \pmod 2$. Similarly, $E(S, G_2) \equiv 0 \pmod 2$. Since every $(4, g)$-cage is edge-superconnected, we need only to discuss three cases for the cutsets S shown in Fig. 6. Cases (a) and (b) are impossible by Lemmas 5 and 6. For (c), we can simply delete edge $s_1 s_2$ from $G[S]$ and obtain a contradiction as in Lemma 5. □

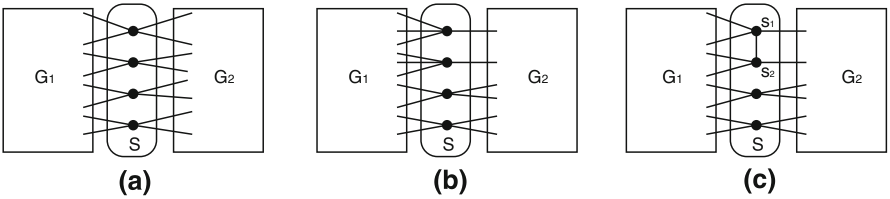

Fig. 6 The three considered cutsets in the proof of Theorem 3

Corollary 2 *Every* $(4, g)$*-cage with odd girth* $g \geq 11$ *is tightly superconnected.*

Proof By contradiction. Let G be a $(4, g)$-cage and S be a 4-cutset such that $G - S$ contains three or more components, say $C_1, C_2, C_3, \ldots$. By Theorem 3, S is the neighborhood of some vertex, that is, $G - S$ contains an isolated vertex, say $V(C_1) = \{v\}$. If $G - S$ contains two isolated vertices, then we have $g(G) = 4$, a contradiction. So $|C_i| \geq 2$ $(i = 2, 3, \ldots)$. Furthermore, since $g \geq 11$, we see $|C_i| \geq 3$ $(i = 2, 3, \ldots)$. Denote $N(v) = \{u_1, u_2, u_3, u_4\}$. Since G is edge-superconnected, so $e(N(v), C_2) \geq 6$ and $e(N(v), C_3) \geq 6$. Note that $d_G(u_j) = 4$, for $j = 1, 2, 3, 4$, so $e(N(v), C_2) = 6$, $e(N(v), C_3) = 6$ and $e(N(v), N(v)) = 0$ and hence $(N(u_1) - v) \cap (V(C_2) \cup V(C_3)) = 3$. So either $|(N(u_1) - v) \cap V(C_2)| = 1$ or $|(N(u_1) - v) \cap V(C_3)| = 1$. Suppose $|(N(u_1) - v) \cap V(C_2)| = 1$ and let $(N(u_1) - v) \cap V(C_2) = \{x\}$, then $S' = \{x, u_2, u_3, u_4\}$ is non-trivial 4-cutset, a contradiction. □

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References

1. Bollobás, B.: Extremal Graph Theory. Academic Press, London (1978)
2. Daven, M.D., Rodger, C.A.: (k, g)-cages are 3-connected. Discrete Math. **199**, 207–215 (1999)
3. Erdös, P., Sachs, H.: Regulare Graphen gegebener Taillenweite mit minimaler Knotenzahl. Wiss. Z. Uni. Halle (Math. Nat.) **12**, 251–257 (1963)
4. Exoo, G., Jajcay, R.: Dynamic cage survey. Electron. J. Combin. **18**, DS16 (2008)
5. Fu, H., Huang, K., Rodger, C.A.: Connectivity of cages. J. Graph Theory **24**, 187–191 (1997)
6. Jiang, T., Mubayi, D.: Connectivity and separating sets of cages. J. Graph Theory **29**, 35–44 (1998)
7. Lin, Y., Miller, M., Balbuena, C., Marcote, X.: All (k, g)-cages are edge-superconnected. Networks **47**, 102–110 (2006)
8. Lin, Y., Miller, M., Balbuena, C.: Improved lower bound for the vertex connectivity of (δ, g)-cages. Discrete Math. **299**, 162–171 (2005)
9. Lin, Y., Miller, M., Rodger, C.A.: All (k, g)-cages are k-edge-connected. J. Graph Theory **48**, 219–227 (2005)
10. Marcote, X., Balbuena, C.: Edge-superconnectivity of cages. Networks **43**, 54–59 (2004)
11. Marcote, X., Pelayo, I., Balbuena, C.: Every cubic cage is quasi 4-connected. Discrete Math. **266**, 311–320 (2003)
12. Marcote, X., Balbuena, C., Pelayo, I., Fàbrega, J.: (δ, g)-cages with $g \geq 10$ are 4-connected. Discrete Math. **301**, 124–136 (2005)
13. Tang, J., Balbuena, C., Lin, Y., Miller, M.: An open problem: $(4, g)$-cage with odd $g \geq 5$ are tightly superconnected. In: Proceedings of the Thirteenth Australasian Symposium on Theory of Computing, vol. 65, pp. 141–144 (2007)
14. Tutte, W.T.: A family of cubical graphs. Proc. Camb. Philos. Soc. **43**, 459–474 (1947)

15. Wang, P., Xu, B., Wang, J.: A note on the edge-connectivity of cages. Electron. J. Combin. **10**, N4 (2003)
16. Wong, P.K.: Cages—a survey. J. Graph Theory **6**, 1–22 (1982)
17. Xu, B., Wang, P., Wang, J.: On the connectivity of (4; g)-cage. Ars Combin. **64**, 181–192 (2002)