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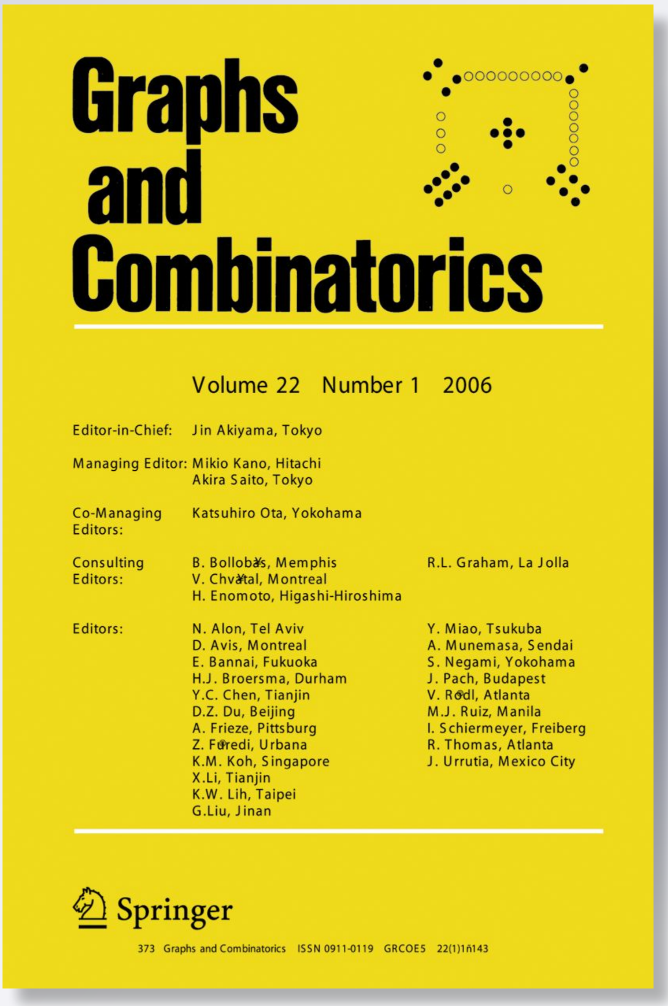

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ORIGINAL PAPER

On Cartesian Product of Factor-Critical Graphs

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Abstract A graph G is *k-factor-critical* if $G - S$ has a perfect matching for any k-subset S of $V(G)$. In this paper, we investigate the factor-criticality in Cartesian products of graphs and show that Cartesian product of an m-factor-critical graph and an n-factor-critical graph is $(m + n + \varepsilon)$-factor-critical, where $\varepsilon = 0$ if both of m and n are even; $\varepsilon = 1$, otherwise. Moreover, this result is best possible.

Keywords Matching · Factor-criticality · Projection · Cartesian product of graphs

1 Introduction

Graphs considered in this paper will be finite, undirected, simple and connected. We use $[x]_2$ to denote the largest even integer not greater than x, i.e., $[x]_2 = 2\lfloor x/2 \rfloor$.

A *perfect matching* is a set of independent edges incident with every vertex of G. A graph G is *k-factor-critical* if $G - S$ has a perfect matching for any k-subset S of $V(G)$. In particular, 0-factor-critical means there exists a perfect matching in G. By definition, we see that a k-factor-critical graph has a perfect matching if and only if k is even and $|V(G)| \geqslant k + 2$. For the cases of $k = 1, 2$, they are also referred as *factor-critical* and *bicritical* graphs by Gallai and Lovász (see [7]), respectively. The factor-critical graphs are used as essential "building blocks" for the so-called

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Gallai-Edmonds matching structure of general graphs and bicritical graphs are studied by Lovász to develop the brick-decomposition as a powerful tool to determine the dimensions of matching lattices.

If every matching of size k can be extended to a perfect matching in G, then G is called *k-extendable*. To avoid triviality, we require that $|V(G)| \geqslant 2k+2$ for k-extendable graphs. This family of graphs was introduced by Plummer in 1980 and studied extensively by Lovász and Plummer [7].

It is natural to study factor criticality and matching extendability of different types of graph products, as such products contain a large number of perfect matchings. Motivation is also from the study of Cayley graphs since graph products often form a 'frame' of Cayley graphs. Győri and Plummer [3] showed that the Cartesian product of an m-extendable graph and an n-extendable graph is $(m+n+1)$-extendable. Győri and Imrich [4] proved that the strong product of an m-extendable graph and an n-extendable graph is $[(m+1)(n+1)]_2$-factor-critical. In the same paper, Győri and Imrich conjectured that the factor-criticality of strong product can be improved to $[(m+2)(n+2)]_2 - 2$. Liu and Yu [6] studied matching extension properties in Cartesian products and lexicographic products. More researches on graph products can be found in the monograph by Imrich and Klavžar [5].

Favaron [2] and Yu [8] introduced the concept of k-factor-critical, independently, and studied the basic properties of k-factor-critical graphs. Several of these properties are used in our proofs, so we summarize them below.

Theorem 1.1 [2,8] *Let G be a k-factor-critical graph with* $k \geqslant 1$, *then*

(1) *G is also* $(k-2)$*-factor-critical if* $k \geqslant 2$;
(2) *G is k-connected*;
(3) *G is* $(k+1)$*-edge-connected. In particular,* $\delta \geqslant k+1$.

In this paper, we investigate the factor-criticality in Cartesian product of an m-factor-critical and an n-factor-critical graphs.

Cartesian product $G_1 \Box G_2$ of two graphs G_1 and G_2 has vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2 or $v_1 = v_2$ and u_1 is adjacent to u_2 in G_1. For example, $K_2 \Box K_2 = C_4$.

For a fixed vertex $v_0 \in V(G_2)$, the *projection* of G_1 in v_0, denoted by $G_1^{v_0}$, is the subgraph of $G_1 \Box G_2$ induced by the vertex set $\{(u, v_0) \mid u \in V(G_1)\}$ and it is called a *row* of $G_1 \Box G_2$. We denote by $G_1^{V_0}$ the subgraph of $G_1 \Box G_2$ induced by the vertex set $\{(u, v) \mid u \in V(G_1), v \in V_0 \subseteq V(G_2)\}$. Similarly, we can define $G_2^{u_0}$ (a *column* of $G_1 \Box G_2$) and $G_2^{U_0}$. Clearly, $G_1^{u_0} \cong G_1$ and $G_2^{v_0} \cong G_2$.

The *projection of a vertical edge* $e = (u, v_1)(u, v_2)$ on G_1^x, where $u \in G_1$ and $v_1, v_2 \in G_2$, denoted by $Proj_{G_1^x}(e)$, is the vertex (u, x) in G_1^x. Similarly, we define $Proj_{G_2^y}(e)$, where $e = (u_1, v)(u_2, v)$ is a horizontal edge. The projections of a vertex $v_0 = (u, v)$ on $G_1^{v_i}$ and $G_2^{u_j}$ are $Proj_{G_1^{v_i}}(v_0) = (u, v_i)$, $Proj_{G_2^{u_j}}(v_0) = (u_j, v)$, respectively.

For terminology and notation not defined here, readers are referred to [1] and [7].

2 Main results

The main result of this paper is the following theorem.

Theorem 2.1 *Let G_1 be an m-factor-critical graph and G_2 an n-factor-critical graph. Then $G_1 \Box G_2$ is $(m+n+\varepsilon)$-factor-critical, where $\varepsilon = 0$, if both of m and n are even; $\varepsilon = 1$, otherwise.*

An interesting special case of Theorem 2.1 is the following theorem. In fact, it will serve as one of basic tools in the proof of Theorem 2.1, so we prove it first.

Theorem 2.2 *Let G be an m-factor-critical graph. Then $G \Box K_2$ is $[m+1]_2$-factor-critical.*

Proof Suppose that G is m-factor-critical, and $V(K_2) = \{v_1, v_2\}$. Let X be a vertex set of $G \Box K_2$ with $|X| = [m+1]_2$.

Case 1. $|X \cap V(G^{v_1})| \equiv |X \cap V(G^{v_2})| \equiv m \pmod 2$.

By the definition of m-factor-criticality and Theorem 1.1, $G^{v_1} - X$ and $G^{v_2} - X$ have perfect matchings M_1 and M_2, respectively. Therefore, $M_1 \cup M_2$ is a perfect matching of $G \Box K_2 - X$.

Case 2. $|X \cap V(G^{v_1})| \equiv |X \cap V(G^{v_2})| \equiv m+1 \pmod 2$.

If $|X \cap V(G^{v_1})|, |X \cap V(G^{v_2})| \leqslant m$, since G is m-factor-critical and hence $|V(G)| \geqslant m+2$, then we can always find a vertical edge uu' between G^{v_1} and G^{v_2} such that both u and u' are not covered by X. So, both $G^{v_1} - X - \{u, u'\}$ and $G^{v_2} - X - \{u, u'\}$ have perfect matchings M_1 and M_2, respectively, as $|(X \cup \{u, u'\}) \cap V(G^{v_i})| \equiv m \pmod 2$ and is at most m for $i = 1, 2$. Therefore, $M_1 \cup M_2 \cup \{uu'\}$ is a perfect matching of $G \Box K_2 - X$.

Without loss of generality, assume $|X \cap V(G^{v_1})| = m+1$ and so m is odd. Select $u \in X$ and so $G^{v_1} - (X - \{u\})$ has a perfect matching M_1. Suppose vv' is the vertical edge of $G \Box K_2$ with $uv \in M_1$ and $v' \in V(G^{v_2})$. Thus, $G^{v_2} - v'$ has a perfect matching M_2, and $(M_1 - uv) \cup M_2 \cup \{vv'\}$ is a perfect matching of $G \Box K_2 - X$. □

In addition, we also need the following lemmas.

Lemma 2.3 *Let G_1 be an m-factor-critical graph and G_2 an n-factor-critical graph with $m, n \geqslant 1$ and n even. If there is an edge $v_1v_2 \in G_2$ such that each component of $G_1 \Box (G_2 - \{v_1, v_2\})$ is $[m+n-1]_2$-factor-critical, then after deleting of any k vertices of $G_1^{\{v_1,v_2\}}$ and any $[m+n+1]_2 - k$ vertices of $G_1 \Box (G_2 - \{v_1, v_2\})$ with $[m+1]_2 + 1 \leqslant k \leqslant [m+n+1]_2$, the remaining subgraph of $G_1 \Box G_2$ has a perfect matching.*

Proof Suppose that there exists an edge $v_1v_2 \in E(G_2)$ such that each component of $G_1 \Box (G_2 - \{v_1, v_2\})$ is $[m+n-1]_2$-factor-critical. Let X be any set of $[m+n+1]_2$ vertices of $G_1 \Box G_2$ with $k = |X \cap V(G_1^{\{v_1,v_2\}})| \geqslant [m+1]_2 + 1$. Let $C_1, \ldots, C_l$ be the connected components of $G_2 - \{v_1, v_2\}$. (Here, l allows to be 1. Moreover, if $l > 1$, as n is even, it follows from Theorem 1.1 that G_2 must be bicritical, $n = 2$ and $[m+1]_2 + 1 \leqslant k \leqslant [m+1]_2 + 2$.)

Choose any $[m+1]_2$-set $X_1 \subseteq X \cap V(G_1^{\{v_1,v_2\}})$, then $G_1^{\{v_1,v_2\}} - X_1$ has a perfect matching M_0 by Theorem 2.2. Consider the edges $x_1y_1, \ldots, x_py_p$ of M_0 such that $x_i \in (X - X_1)$ and $y_i \notin (X - X_1)$. Note that $p \leqslant k - [m+1]_2 \leqslant n$.

Case 1. $l = 1$.

As G_2 is n-factor-critical, G_2 is $(n+1)$-edge-connected and $\delta(G_2) \geqslant n+1$ by Theorem 1.1 (3). Since $l = 1$, both of v_1 and v_2 have at least n neighbors in C_1, then each $y_i (1 \leqslant i \leqslant p)$ has at least n neighbors in $G_1 \Box C_1$. Now, $|X \cap V(G_1 \Box C_1)| = [m+n+1]_2 - |X \cap V(G_1^{\{v_1,v_2\}})| \leqslant n - p$, so we can always find distinct vertices $z_1, \ldots, z_p$ in $V(G_1 \Box C_1) - X$ such that $y_i z_i \in E(G_1 \Box G_2)$ and $|(X \cup \{z_1, \ldots, z_p\}) \cap V(G_1 \Box C_1)| \equiv 0 \pmod 2$. Since $|(X \cup \{z_1, \ldots, z_p\}) \cap V(G_1 \Box C_1)| \leqslant [m+n+1]_2 - ([m+1]_2 + p) + p \leqslant [m+n-1]_2$, by assumption, $G_1 \Box C_1 - (X \cup \{z_1, \ldots, z_p\})$ has a perfect matching M_1. Let M_0' denote the set of edges of M_0 with both ends in X. Then $M_0 \cup M_1 \cup \{y_1z_1, \ldots, y_pz_p\} - M_0' - \{x_1y_1, \ldots, x_py_p\}$ is a perfect matching of $G_1 \Box G_2 - X$.

Case 2. $l > 1$, $k = [m+1]_2 + 2$.

In this case $n = 2$, $X \subseteq V(G_1^{\{v_1,v_2\}})$ and p equals to either 0 or 2. If $p = 0$, as G_1 and C_j $(1 \leqslant j \leqslant l)$ are m-factor-critical and 0-factor-critical, respectively, there exists a perfect matching M_j in $G_1 \Box C_j$. Then $\bigcup_{j=0}^{l} M_j - \{e_0\}$ is a perfect matching of $G_1 \Box G_2 - X$, where e_0 denotes the edge of M_0 with both ends in X. If $p = 2$ and $y_1y_2 \in E(G_1^{\{v_1,v_2\}})$, let M_j denote a perfect matching of $G_1 \Box C_j$ for all $1 \leqslant j \leqslant l$, then $\bigcup_{j=0}^{l} M_j \cup \{y_1y_2\} - \{x_1y_1, x_2y_2\}$ is a perfect matching of $G_1 \Box G_2 - X$. At last, assume $p = 2$ and $y_1y_2 \notin E(G_1^{\{v_1,v_2\}})$. Since G_2 is 3-edge-connected, both v_1 and v_2 are adjacent to each C_j. Hence, we can match y_1 and y_2 with two vertices z_1, z_2 in $G_1 \Box C_j$ such that $y_iz_i \in E(G_1 \Box G_2 - X)$ and $|(X \cup \{z_1, z_2\}) \cap V(G_1 \Box C_j)| \equiv 0 \pmod 2$ for all $1 \leqslant j \leqslant l$. Since $|(X \cup \{z_1, z_2\}) \cap V(G_1 \Box C_j)| \leqslant 2 \leqslant [m+n-1]_2$, by assumption, $G_1 \Box C_j - (X \cup \{z_1, z_2\})$ has a perfect matching M_j for all $1 \leqslant j \leqslant l$. Therefore, $\bigcup_{j=0}^{l} M_j \cup \{y_1z_1, y_2z_2\} - \{x_1y_1, x_2y_2\}$ is a perfect matching of $G_1 \Box G_2 - X$.

Case 3. $l > 1$ and $k = [m+1]_2 + 1$.

Now $n = 2$ and $p = 1$, there exists only one component, say C_1, satisfying $X \cap V(G_1 \Box C_1) \neq \emptyset$. Furthermore, $|X \cap V(G_1 \Box C_1)| = |X| - k = 1$. Assume $(u_0, v_0) \in X \cap V(G_1 \Box C_1)$. Since G_2 is bicritical, it is 2-connected and 3-edge-connected. So every v_i $(i = 1, 2)$ has at least one neighbor in C_j for all j $(1 \leqslant j \leqslant l)$. Then y_1 has at least one neighbor in $G_1 \Box C_j$ for all j $(1 \leqslant j \leqslant l)$. There are two subcases to consider.

Subcase 3.1. y_1 has a neighbor z_1 in $G_1 \Box C_1 - X$.

Clearly, $|(X \cup \{z_1\}) \cap V(G_1 \Box C_j)|$ equals to 0 or 2 for each $1 \leqslant j \leqslant l$. But $[m+n-1]_2 \geqslant 2$, by assumption, $G_1 \Box C_j - (X \cup \{z_1\})$ has a perfect matching M_j for all j $(1 \leqslant j \leqslant l)$ and thus $\bigcup_{j=0}^{l} M_j \cup \{y_1z_1\} - \{x_1y_1\}$ is a perfect matching of $G_1 \Box G_2 - X$.

Subcase 3.2. y_1 doesn't have any neighbor in $G_1 \Box C_1 - X$.

So y_1 is adjacent to (u_0, v_0). Since $d_{G_1 \Box G_2}(y_1) \geqslant m+1+n+1 > [m+n+1]_2$, there exists a vertex $z_1 \in V(G_1 \Box G_2) - X$ such that $y_1z_1 \in E(G_1 \Box G_2)$.

If $z_1 \in V(G_1^{\{v_1,v_2\}})$, we may assume z_1 is matched with z_1' in M_0. It is not difficult to see that z_1' is not adjacent to (u_0, v_0). As G_2 is 3-edge-connected, z_1' has a neighbor

z_2 in $G_1\Box C_1-(u_0,v_0)$. Then $|(X\cup\{z_1,z_2\})\cap V(G_1\Box C_j)|$ equals to 0 or 2, and hence $G_1\Box C_j-(X\cup\{z_1,z_2\})$ has a perfect matching M_j for $j(1\leqslant j\leqslant l)$. Thus, $\bigcup_{j=0}^{l}M_j\cup\{y_1z_1,z_1'z_2\}-\{x_1y_1,z_1z_1'\}$ is a perfect matching of $G_1\Box G_2-X$.

Without loss of generality, suppose $z_1\in V(G_1\Box C_2)$ and y_1 contains no neighbor in $G_1^{\{v_1,v_2\}}-X$. Since $|V(G_1^{\{v_1,v_2\}})|\geqslant 2(m+2)$, there must be an edge $z_2z_3\in M_0$. Note that y_1 doesn't in the same column with both z_2 and z_3; neither do z_1 and (u_0,v_0). Because G_2 is 3-edge-connected, we can find $z_2'\in V(G_1\Box C_1)$, $z_3'\in V(G_1\Box C_2)$ such that $z_iz_i'\in E(G_1\Box G_2-(X\cup\{z_1\}))$. Since $|(X\cup\{z_1,z_2',z_3'\})\cap V(G_1\Box C_j)|$ equals to 0 or 2, $G_1\Box C_j-(X\cup\{z_1,z_2',z_3'\})$ has a perfect matching M_j for all j $(1\leqslant j\leqslant l)$, and hence $\bigcup_{j=0}^{l}M_j\cup\{y_1z_1,z_2z_2',z_3z_3'\}-\{x_1y_1,z_2z_3\}$ is the desired perfect matching of $G_1\Box G_2-X$.

We complete the proof. □

Use the same technique, we can prove the following result about factor-criticality when mn is odd.

Lemma 2.4 *Let G_1 be m-factor-critical, and G_2 n-factor-critical with $m,n\geqslant 1$ and mn odd. If there is an edge $v_1v_2\in G_2$ such that each component of $G_1\Box(G_2-\{v_1,v_2\})$ is $(m+n-1)$-factor-critical, then after deletion of any k vertices of $G_1^{\{v_1,v_2\}}$ and any $m+n+1-k$ vertices of $G_1\Box(G_2-\{v_1,v_2\})$ with $m+2\leqslant k\leqslant m+n+1$, the remaining subgraph of $G_1\Box G_2$ has a perfect matching.*

Lemma 2.5 *Let G_1 be m-factor-critical and G_2 n-factor-critical, where m,n are positive even integers. Let X be an arbitrary subset of $V(G_1\Box G_2)$ with $|X|=m+n$. If for any $u_iu_j\in E(G_1)$, $|X\cap V(G_2^{\{u_i,u_j\}})|\leqslant 1$ and for any $v_iv_j\in E(G_2)$, $|X\cap V(G_1^{\{v_i,v_j\}})|\leqslant 1$, then there exists a perfect matching in $G_1\Box G_2-X$.*

Proof Without loss of generality, assume that $m\geqslant n$ and $|V(G_2)|=2t$. Let $I:=\{v_i\mid v_i\in V(G_2),\ |X\cap V(G_1^{v_i})|=1\}$. Then $|I|=m+n$ and it is an independent set of G_2.

Since G_2 is n-factor-critical with n even, there is a perfect matching in G_2 and $|V(G_2)|\geqslant 2(m+n)$. (Note that any vertex in I must be matched with a vertex in G_2-I.) Furthermore, for any n-vertex set $N\subseteq V(G_2)-I$, there is a perfect matching in G_2-N and $|V(G_2)-N|\geqslant 2(m+n)$. Therefore, $|V(G_2)|\geqslant n+2(m+n)$. Similarly, $|V(G_1)|\geqslant m+2(m+n)$. Since G_2 is bicritical, it is non-bipartite and then there exists an edge, say $e=v_1v_2\in E(G_2-I)$, of G_2 such that $|X\cap V(G_1^{\{v_1,v_2\}})|=0$. Now we relabel the vertices of G_2 as an ordered sequence $v_1',v_2',v_3',\ldots$, where $v_1'=v_1$ and $v_2'=v_2$, satisfying the following property

$$\text{each}\,v_i' \text{ has at least one neighbor in } \{v_1',\ldots,v_{i-1}'\}. \tag{*}$$

Such a sequence can be easily constructed. For example, find a spanning tree T of G_2 with root v_1, and put the vertices with distance 1 to v_1 in the sequence first (in arbitrary order except v_2), then vertices with distance 2, …, to obtain a desired sequence (see Example 2.6). So we obtain an ordering of $V(G_2)$, and thus an ordering of rows of $G_1\Box G_2$. For convenience, denote this ordering of rows by $\mathbb{S}: G_1^{v'_{2t}}, G_1^{v'_{2t-1}}, \ldots, G_1^{v'_2}, G_1^{v'_1}$.

We start our proof by describing a Transitive Projection Method (TPM). Without loss of generality, assume $\mathbb{S} := G_1^{v_{2t}}, G_1^{v_{2t-1}}, \ldots, G_1^{v_2}, G_1^{v_1}$ from now on.

The aim of this method is to find a matching M such that $|(X \cup V_M) \cap V(G_1^v)|$ is even, for every $v \in V(G_2)$. (Hereafter, let V_M denote the vertex set of the graph induced by M in $G_1\Box G_2$.) The matching M consists of vertical edges of $G_1\Box G_2$, which is constructed step by step.

TPM. Set $M = \emptyset$ and process each row $G_1^{v_i}$ according to the order $\mathbb{S}$. At first, consider $G_1^{v_{2t}}$ and set $k := 2t$.

Step 1. If $k = 1$, stop.

If $k > 1$ and

$$|X \cap V(G_1^{v_k})| + |V_M \cap V(G_1^{v_k})| \leqslant m,$$

then go to Step 2.

If $k > 1$ and

$$|X \cap V(G_1^{v_k})| + |V_M \cap V(G_1^{v_k})| > m, \tag{**}$$

then go to Step 4.

Step 2. If $|X \cap V(G_1^{v_k})| + |V_M \cap V(G_1^{v_k})|$ is even (0 is allowed), then set $k := k - 1$ and go to Step 1; otherwise go to Step 3.

Step 3. Find the first neighbor v of v_k ('first' means that $v \in \{v_{k-1}, \ldots, v_1\}$, and if $v_k v_i \in E(G_2)$ and $v_k v_j \in E(G_2)$ with $k > i > j$, then we prefer v_i over v_j). Similarly, we can find the first neighbor v' of v. (Note that if $|X \cap V(G_1^{v'})| = 1$, denote the common vertex by (x, v'), then $\{(x, v), (x, v_k)\} \cap X = \emptyset$ and $|X \cap V(G_2^x)| \leqslant 1$ as $|X \cap V(G_2^{\{u_i,u_j\}})| \leqslant 1$ for any $u_i u_j \in E(G_1)$.) We consider three cases:

(1) If $|X \cap V(G_1^{v'})| = 0$ (or 1 and $(x, v) \in V_M$), then find a vertical edge e between $G_1^{v_k}$ and G_1^v, with both ends being not covered by $X \cup V_M$, and add it to M. Set $k := k - 1$ and go to Step 1.

(2) If $|X \cap V(G_1^{v'})| = 1$ and $(x, v) \notin V_M$, $(x, v_k) \notin V_M$, then set $e := (x, v_k)(x, v)$ and add it to M. Set $k := k - 1$ and go to Step 1.

(3) If $|X \cap V(G_1^{v'})| = 1$ and $(x, v) \notin V_M$, then $(x, v_k) \in V_M$, we may assume that (x, v_k) is matched with (x, v_i) (where $i > k$) under M and then replace the vertical edge $(x, v_i)(x, v_k)$ by another vertical edge e' between $G_1^{v_i}$ and $G_1^{v_k}$ such that both ends of e' are not covered by $X \cup V_M$. Set $e := (x, v_k)(x, v)$, $M := M \cup \{e, e'\} - (x, v_i)(x, v_k)$, $k := k - 1$, and go to Step 1.

Step 4. Suppose that $|X \cap V(G_1^{v_k})| + |V_M \cap V(G_1^{v_k})| = m + l$, by the construction of M, then $1 \leqslant l \leqslant n \leqslant m$. Denote the vertex v_k by v^*, and assume that v is the first neighbor of v^*. Select m vertices from $(X \cup V_M) \cap V(G_1^{v^*})$, denote the set of selected vertices by X^*, such that if $X \cap V(G_1^v) = \{(x, v)\}$, then $(x, v^*) \in X^*$. This is possible according to the construction of M in Step 3.

Clearly, $G_1^{v^*} - X^*$ has a perfect matching M^*. Consider the edges $e_i = y_i z_i$ $(1 \leqslant i \leqslant p)$ of M^* such that $y_i \notin X - X^*$ and $z_i \in X - X^*$. Then $p \leqslant l$ and $p \equiv l \pmod 2$. For each y_i $(1 \leqslant i \leqslant p)$,

(1) if $y'_i = Proj_{G_1^v}(y_i) \notin (X \cup V_M)$, then add $y_i y'_i$ to M and set $M^* := M^* - e_i$;
(2) if $y'_i = Proj_{G_1^v}(y_i) \in (X \cup V_M)$, say $y'_i w_i \in M$, then replace $y'_i w_i$ by another vertical edge e' such that both ends of e' are not covered by V_M. Here vertical edges $y'_i w_i$ and e' are between two same rows. Set $e := y_i y'_i$, $M := M \cup \{e, e'\} - y'_i w_i$ and $M^* := M^* - e_i$.
Finally, set $k := k - 1$ and go to Step 1. (See Example 2.6 for an illustration.)

To insure the validity of TPM, we need to verify the following:

(1) The above method is feasible;
(2) the case (**) occurs at most once;
(3) $|X \cap V(G_1^{v_i})| + |V_M \cap V(G_1^{v_i})|$ is even and less than m for all $v_i \in \{v_{2t}, \ldots, v_2\}$ except for v^* if (**) occurs.
(4) $|X \cap V(G_1^{v_1})| + |V_M \cap V(G_1^{v_1})|$ is even and no more than m.

By the construction of M, the assertion (3) holds. It is not difficult to see that the process of constructing M is actually to pass the vertices common with X from one row to another row. So, as $|X| = m + n (m \geqslant n)$, (**) occurs at most once, that is, the assertion (2) holds. Furthermore, since $|V(G_1)| \geqslant 3m + 2n$, so $|V(G_1)| > 2|X| > m + 2n \geqslant m + 2l$ and thus the above method is always feasible.

It remains to confirm the assertion (4). If (**) doesn't occur, then (4) holds as $|(X \cup V_M) \cap V(G_1^{v_i})|$ $(i \neq 1)$ and $|X \cup V_M|$ are even. If (**) occurs, then (4) holds because $p \equiv l \pmod 2$, and $|X \cap V(G_1^{v_1})| + |V_M \cap V(G_1^{v_1})|$ has the same parity as $|X \cup V_M| - (m + l) + p$ from the construction of M.

Therefore $G_1^{v_i} - X \cup V_M$ has a perfect matching M_i for each $v_i \in \{v_{2t}, \ldots, v_1\}$ with $v_i \neq v^*$ by Theorem 1.1.

Let M_0 be the edge set of M^* with both ends in X if (**) occurs. Then, when (**) occurs, $G_1 \Box G_2 - X$ has a perfect matching $\bigcup_{i=1, v_i \neq v^*}^{2t} M_i \cup (M^* - M_0) \cup M$. Otherwise, $G_1 \Box G_2 - X$ has a perfect matching $\bigcup_{i=1}^{2t} M_i \cup M$. □

Example 2.6 Let G_1 and G_2 be two bicritical graphs shown in Fig. 1, where $m = n = 2$. Suppose $X = \{(u_1, v_3), (u_3, v_5), (u_6, v_9), (u_8, v_{10})\}$. Clearly, G_1, G_2, X satisfy the conditions of Lemma 2.5. To find a perfect matching by TPM, starting with v_2, we find an ordering of $G_2 - v_1$ satisfying property (*), by neighborhood relations ($\rightarrow$) as following:

$$v_2(v'_2) \rightarrow \left\{ \begin{array}{l} v_3(v'_3) \rightarrow v_7(v'_5) \rightarrow \left\{ \begin{array}{l} v_6(v'_7) \rightarrow \left\{ \begin{array}{l} v_4(v'_{10}) \\ v_5(v'_{11}) \end{array} \right. \\ v_{10}(v'_8) \rightarrow v_{11}(v'_{12}) \end{array} \right. \\ v_8(v'_4) \rightarrow v_9(v'_6) \rightarrow v_{12}(v'_9) \end{array} \right. \tag{3}$$

Hence a sequence is

$$\begin{aligned} \mathbb{S} &= G_1^{v'_{12}}, G_1^{v'_{11}}, \ldots, G_1^{v'_1} \\ &= G_1^{v_{11}}, G_1^{v_5}, G_1^{v_4}, \ldots, G_1^{v_1}. \end{aligned}$$

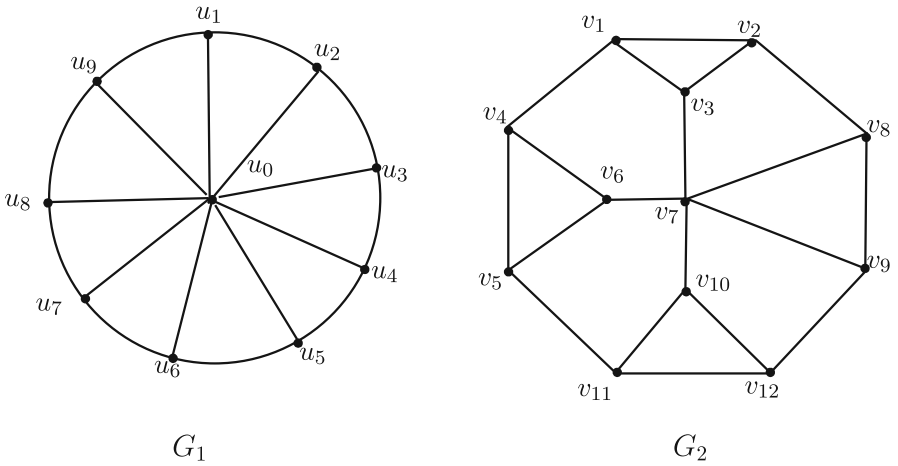


Fig. 1 Two bicritical graphs G_1 and G_2

Next, we construct a matching M by TPM:

When $k = 12$, then $G_1^{v'_{12}} = G_1^{v_{11}}$ and $M = \emptyset$;

when $k = 11$, then $G_1^{v'_{11}} = G_1^{v_5}$ and $e_1 = (u_6, v_5)(u_6, v_4)$;

when $k = 10$, then $G_1^{v'_{10}} = G_1^{v_4}$ and $e_2 = (u_5, v_4)(u_5, v_6)$;

when $k = 9$, then $G_1^{v'_9} = G_1^{v_{12}}$ and no edge is selected and $M := M$; continue on, we obtain edges $e_3 = (u_1, v_{10})(u_1, v_7)$, $e_4 = (u_4, v_6)(u_4, v_7)$ and $e_5 = (u_5, v_9)(u_5, v_7)$;

when $k = 5$, then $G_1^{v'_5} = G_1^{v_7}$ and $|X \cap V(G_1^{v_7})| + |V_M \cap V(G_1^{v_7})| = 3 > 2 = m$. We select $X^* = \{(u_1, v_7), (u_4, v_7)\}$ and thus $G_1^{v_7} - X^*$ has a perfect matching $\{(u_2, v_7)(u_3, v_7), (u_5, v_7)(u_6, v_7), (u_7, v_7)(u_8, v_7), (u_9, v_7)(u_0, v_7)\}$. So, set $e_6 = (u_6, v_7)(u_6, v_8)$; similarly, we have $e_7 = (u_9, v_8)(u_9, v_2)$, $e_8 = (u_0, v_3)(u_0, v_2)$.

At the end, we obtain a matching $M = \{e_1, e_2, \ldots, e_8\}$ such that $|(X \cup V_M) \cap V(G_1^v)|$ is even for every $v \in V(G_2)$ (see Fig. 2).

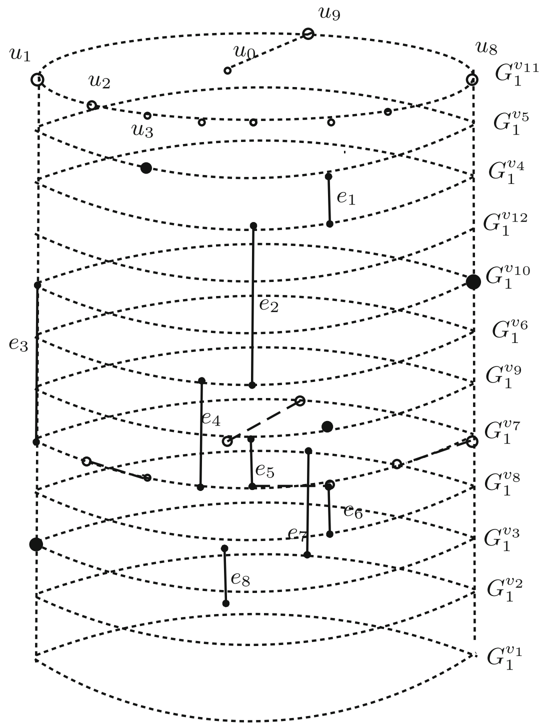
3 Proofs of the main results

Before proving Theorem 2.1, we introduce two algorithms which find specific matchings in two special cases.

Suppose that G_1 is m-factor-critical and G_2 is 0-factor-critical and connected (resp. G_1 is m-factor-critical with m odd and G_2 is 1-factor-critical). Let X be any subset of $V(G_1 \square G_2)$ with $|X| = [m+1]_2$ if $n = 0$ (resp. $m + 2$ if $n = 1$). If $n = 0$ (resp. $n = 1$), G_2 has a perfect matching $\{v_1v_2, \ldots, v_{2t-1}v_{2t}\}$ (resp. $G_2 - v$ has a perfect matching $\{v_1v_2, \ldots, v_{2t-1}v_{2t}\}$, where v satisfies $|X \cap V(G_1^v)| \equiv m \pmod 2$ and $|X \cap V(G_1^v)| > 0$). Suppose $|X \cap V(G_1^{\{v_{2i-1}, v_{2i}\}})|$ is odd for some i $(1 \leqslant i \leqslant t)$ and I_0 denotes the set of such indices i. Clearly, $|I_0|$ is even.

We construct a matching M consisting of vertical edges of $G_1 \square G_2$ step by step and satisfying

Fig. 2 Finding M in Example 2.6



(1) $X \cap V_M = \emptyset$;

(2) $|X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| + |V_M \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| \leqslant [m+1]_2$ is even for all $i (1 \leqslant i \leqslant t)$.

Algorithm A_1

Starting with $M = \emptyset$, $I := I_0$.

Step 1. Choose any $i_0, j_0 \in I$ and find a path P in G_2 from v_{2i_0-1} (or v_{2i_0}) to v_{2j_0-1} (or v_{2j_0}). This is possible as G_2 is connected.

Step 2. For each edge $e = xy$ in P,

(a) if $e \neq v_{2i-1}v_{2i}$ for each i $(1 \leqslant i \leqslant t)$, then choose a vertical edge e' between G_1^x and G_1^y such that both endvertices of e' are not covered by X and M, set $M := M \cup \{e'\}$;

(b) if $e = v_{2i-1}v_{2i}$ for some i $(1 \leqslant i \leqslant t)$, then set $M := M$;

Step 3. Set $I := I - \{i_0, j_0\}$. If $I = \emptyset$, stop; else, go to Step 1.

To see the validity of Algorithm A_1, note that Step 2 is always possible since

(1) $|V(G_1^{v_i})| \geqslant m+2$, for $v_i \in V(G_2)$;

(2) if $|X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})|$ is even, then $i \notin I$ and

$$|X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| + |V_M \cap V(G_1^{\{v_{2i-1},v_{2i}\}})|$$
$$\leqslant |X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| + 2\frac{[m+1]_2 - |X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})|}{2} = [m+1]_2;$$

(3) if $|X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})|$ is odd, then $i \in I$ and
$|X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| + |V_M \cap V(G_1^{\{v_{2i-1},v_{2i}\}})|$
$\leqslant |X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| + 2\left(\frac{[m+1]_2 - |X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| + 1}{2} - 1\right) + 1$
$= [m+1]_2$;

(4) $|\{u \in V(G_1) | (u,x) \in X \cup V_M \text{ or } (u,y) \in X \cup V_M\}| \leqslant [m+1]_2$ for any $xy \in E(G_2)$ by the construction of M and (2), (3).

Note that if $n = 1$, $|X \cap V(G_1^v)| + |V_M \cap V(G_1^v)| \equiv m \pmod 2$ and is no more than $m+2$. If $m+2$ is reached, every path P constructed must 'pass' through G_1^v, and then for all i $(1 \leqslant i \leqslant t)$, $|X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| \leqslant 1$. If v is not a cut vertex, we can change some paths so that $|X \cap V(G_1^v)| + |V_M \cap V(G_1^v)|$ decreases by at least 2. If v is a cut vertex, there exists $v'(\neq v) \in G_2$ such that $|X \cap V(G_1^{v'})| \equiv m \pmod 2$. Set $v := v'$ and apply Algorithm A_1 again. Above all, we can always find v and a desired M satisfying $m \geqslant |X \cap V(G_1^v)| + |V_M \cap V(G_1^v)| \equiv m \pmod 2$.

Now, suppose that G_1 is m-factor-critical with m even and G_2 is 1-factor-critical. Let X be any subset of $V(G_1 \square G_2)$ with $|X| = m+2$. Suppose $|V(G_2)| = 2t+1$ and $|X \cap V(G_1^{v_i})| \equiv 1 \pmod 2$ for some i $(0 \leqslant i \leqslant 2t)$ and I_0 denotes the set of such indices i. Clearly, $|I_0|$ is even. We would like to construct a matching M of $G_1 \square G_2$ and an induced subgraph F of G_2.

Algorithm A_2

Starting with $F = \emptyset$, $M = \emptyset$, $I := I_0$, $\mathcal{P} = \emptyset$.

Step 1. Choose any $i_0, j_0 \in I$ and find a path P in G_2 from v_{i_0} to v_{j_0}.

Step 2. Set $I := I - \{i_0, j_0\}$, $F := F \triangle P$ ($\triangle$ denotes symmetric difference) and $\mathcal{P} := \mathcal{P} \cup \{P\}$. If there is an Eulerian cycle in F, delete all the edges of the cycle from F. If $I = \emptyset$, stop; else, go to Step 1.

Let $d_F(v)$ denote the degree of v in F. Then $|X \cap V(G_1^v)| + d_F(v) \equiv m \pmod 2$, for each $v \in V(G_2)$. Similar to A_1, we can prove that $|X \cap V(G_1^v)| + d_F(v) \leqslant m+2$. Moreover, if $m+2$ is reached for some v, then each path $P \in \mathcal{P}$ contains v and thus there is at most one such vertex v by construction of F. Choose a row $G_1^{v_0}$ such that $|X \cap V(G_1^{v_0})| + d_F(v_0) = \max\{|X \cap V(G_1^v)| + d_F(v) | v \in V(G_2)\}$. When $|X \cap V(G_1^{v_0})| + d_F(v_0) \leqslant m$, go to Step 3; when $|X \cap V(G_1^{v_0})| + d_F(v_0) = m+2$, go to Step 4.

Step 3. For each edge $e = xy$ in $E(F)$, choose a vertical edge e' between G_1^x and G_1^y such that both end-vertices of e' are not covered by X and M, set $M := M \cup \{e'\}$;

Step 4. When $|X \cap V(G_1^{v_0})| + d_F(v_0) = m+2$, every path we constructed above should 'pass' the row $G_1^{v_0}$, so for all $v \neq v_0$, $|X \cap V(G_1^v)| \leqslant 1$ and $|X \cap V(G_1^v)| + d_F(v) \leqslant 2$.

(1) If we can replace a path $P \in \mathcal{P}$ by another path P' in $G_2 - v$ and P' has the same end-vertices with P, then set $F := (F \triangle P) \triangle P'$ and go to Step 3.
(2) Otherwise, v_0 is a cut vertex of G_2, we stop and set $F = \emptyset$, $M = \emptyset$. Note that in this case, $|X \cap V(G_1 \square C)| \leq 1$ for all connected component C of $G_2 - v_0$.

The validity of Step 3 can be argued in the same way as in Algorithm A_1.

In Step 4, when $m \geqslant 4$, whenever we change a path P, $|X \cap V(G_1^{v_0})| + d_F(v_0)$ decreases by at least 2. Then for any $v \neq v_0$, $|X \cap V(G_1^v)| + d_F(v) \leqslant 4$, and we can go to Step 3. When $m = 2$, we can choose i_0 and j_0 properly to avoid this.

Now, we are ready to prove the main theorem.

3.1 Proof of Theorem 2.1

Suppose that G_1 is m-factor-critical and G_2 is n-factor-critical, where $m \geqslant n$. We use induction on $m + n$.

When $n = 0$, let $M^* = \{v_1v_2, \ldots, v_{2t-1}v_{2t}\}$ be a perfect matching of G_2 and X a vertex set with $|X| = [m+1]_2$.

Case 1. $|X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})|$ is even for all i $(1 \leqslant i \leqslant t)$.

Clearly, $G_1^{\{v_{2i-1},v_{2i}\}} - X$ has a perfect matching M_i by Theorems 1.1 and 2.2. Thus, $\bigcup_{i=1}^t M_i$ is a perfect matching of $G_1 \Box G_2 - X$.

Case 2. $|X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})|$ is odd for some $i (1 \leqslant i \leqslant t)$.

We apply Algorithm A_1 and obtain a matching M such that $|X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| + |V_M \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| \leqslant [m+1]_2$ is even, for all $i (1 \leqslant i \leqslant t)$. Thus, $G_1^{\{v_{2i-1},v_{2i}\}} - (X \cup V_M)$ has a perfect matching M_i by Theorems 2.2 and 1.1.

Hence, $M \cup \bigcup_{i=1}^t M_i$ is a perfect matching of $G_1 \Box G_2 - X$.

When $m \geqslant n = 1$, let $|X| = m + 2$. We consider the following cases.

Case 1. $|X \cap V(G_1^v)| \leqslant m$ for all $v \in V(G_2)$.

Subcase 1.1. m is odd. Then $|X| = m+2$ is odd and there exists a row, say G_1^v, such that $|X \cap V(G_1^v)|$ is odd. So $G_1^v - X$ has a perfect matching M_0 as $|X \cap V(G_1^v)| \leqslant m$ and $|X \cap V(G_1^v)| \equiv m \pmod 2$. On the other hand, G_2 is 1-factor-critical, $G_2 - v$ has a perfect matching $\{v_1v_2, \ldots, v_{2t-1}v_{2t}\}$.

Subcase 1.1.1. $|X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})|$ is even, for all i $(1 \leqslant i \leqslant t)$.

Then $|X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| \leqslant [m+1]_2$ for each i $(1 \leqslant i \leqslant t)$ and thus $G_1^{\{v_{2i-1},v_{2i}\}} - X$ has a perfect matching M_i. So $\bigcup_{i=0}^t M_i$ is a perfect matching of $G_1 \Box G_2 - X$.

Subcase 1.1.2. $|X \cap V(G_1^{\{v_{2i-1}v_{2i}\}})|$ is odd for some i $(1 \leqslant i \leqslant t)$.

We use Algorithm A_1 to obtain a matching M such that $X \cap V_M = \emptyset$ and $|X \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| + |V_M \cap V(G_1^{\{v_{2i-1},v_{2i}\}})| \leqslant [m+1]_2$ is even for all $i (1 \leqslant i \leqslant t)$. Moreover, $|(X \cup V_M) \cap V(G_1^v)| \equiv m \pmod 2$ is less than m. Let M_i and M_0' be perfect matchings of $G_1^{\{v_{2i-1},v_{2i}\}} - (X \cup V_M)$ and $G_1^v - (X \cup V_M)$, respectively. Then $\bigcup_{i=1}^t M_i \cup M_0' \cup M$ is a perfect matching of $G_1 \Box G_2 - X$.

Subcase 1.2. m is even. We apply Algorithm A_2 to obtain a matching M. Suppose $|V(G_2)| = 2t + 1$.

Subcase 1.2.1. If $M \neq \emptyset$, since $|(X \cup V_M) \cap V(G_1^{v_i})| \equiv 0 \pmod 2$ and is less than m, $G_1^{v_i} - (X \cup V_M)$ has a perfect matching M_i for each $v_i \in V(G_2)$. Then $\bigcup_{i=0}^{2t} M_i \cup M$ is a perfect matching of $G_1 \Box G_2 - X$.

Subcase 1.2.2. If $M = \emptyset$, in this case, v_0 is a cut-vertex of G_2. Let $C_1, \ldots, C_l$ be the connected components of $G_2 - v_0$. So $|X \cap V(G_1 \Box C_j)| \leqslant 1$ and $d_{G_1 \Box C_j}(v_0) \geqslant 2$ for $j = 1, \ldots, l$, since G_2 is 1-factor-critical and 2-edge-connected. Assume $|X \cap V(G_1 \Box C_j)| = 1$ for $j = 1, 2, \ldots, p$. Clearly, $p + |X \cap V(G_1^{v_0})| = m+2 \leqslant |V(G_1^{v_0})|$.

If $|X \cap V(G_1^{v_0})| \equiv 0 \pmod 2$, $G_1^{v_0} - X$ has a perfect matching M_0. Note that $p \leqslant 2|M_0|$ is even. Consider the edges $x_1x_2, \ldots, x_{p-1}x_p$ of M_0. For each x_i $(1 \leqslant i \leqslant p)$, it has at least two neighbors in $G_1 \Box C_j$ for all $1 \leqslant j \leqslant p$ as G_2 is 2-edge-connected, we can find y_i in $V(G_1 \Box C_i) - X$ such that $x_i y_i \in E(G_1 \Box G_2)$. Now $|(X \cup \{y_1, \ldots, y_p\}) \cap V(G_1 \Box C_j)| \leqslant 2 \leqslant m$ and is even. Since C_j is 0-factor-critical and $G_1 \Box C_j$ is m-factor-critical for $j = 1, \ldots, l$. So $G_1 \Box C_j - (X \cup \{y_1, \ldots, y_p\})$ has a perfect matching M_j for all j $(1 \leqslant j \leqslant l)$. Then $\bigcup_{j=0}^{l} M_j \cup \{x_1y_1, \ldots, x_py_p\} - \{x_1x_2, \ldots, x_{p-1}x_p\}$ is a perfect matching of $G_1 \Box G_2 - X$.

If $|X \cap V(G_1^{v_0})| \equiv 1 \pmod 2$, we choose a vertex x_0 from $X \cap V(G_1^{v_0})$ and let $X - \{x_0\} = X_1$. So $|X_1|$ is even, $G_1^{v_0} - X_1$ has a perfect matching M_0 with $|M_0| = |V(G_1^{v_0})| - |X_1| \geqslant p + 1$. Suppose x_0 is matched with x_1 in M_0. Consider the edges $x_0x_1, x_2x_3, \ldots, x_{p-1}x_p$ of M_0. For each x_i $(i = 1, 2, \ldots, p)$, it has at least two neighbors in $G_1 \Box C_j$ for each j $(1 \leqslant j \leqslant p)$. So we can find y_i in $V(G_1 \Box C_i) - X$ such that $x_i y_i \in E(G_1 \Box G_2)$. The same as before, $G_1 \Box C_j - (X \cup \{y_1, \ldots, y_p\})$ has a perfect matching M_j for all j $(1 \leqslant j \leqslant l)$. Then $\bigcup_{j=0}^{l} M_j \cup \{x_1y_1, \ldots, x_py_p\} - \{x_0x_1, \ldots, x_{p-1}x_p\}$ is a perfect matching of $G_1 \Box G_2 - X$.

Case 2. $m + 1 \leqslant |X \cap V(G_1^{v})| \leqslant m + 2$ for some $v \in V(G_2)$.

Let $C_1, \ldots, C_l$ (here l allows to be 1) be connected components of $G_2 - v$. Since G_2 is 1-factor-critical, each C_j has a perfect matching. Choose any m-vertex set $X_1 \subseteq X \cap V(G_1^{v})$, then $G_1^{v} - X_1$ has a perfect matching M_0. Consider edges $x_1y_1, \ldots, x_py_p$ $(0 \leqslant p \leqslant 2)$ of M_0 with $x_i \in X - X_1$ and $y_i \in V(G_1^{v}) - X$. If $p = 0$, then $|X \cap V(G_1^{v})| = m + 2$ and $|X \cap V(G_1 \Box C_j)| = 0$. If $p \geqslant 1$, for each y_i $(1 \leqslant i \leqslant p)$, it has at least two neighbors in $G_1 \Box C_j$ for any j $(1 \leqslant j \leqslant l)$ as G_2 is 2-edge-connected. But $|X \cap V(G_1 \Box C_j)| \leqslant (m + 2) - (m + p) \leqslant 1$, so we can find distinct vertices $z_1, \ldots, z_p$ in $G_1 \Box (G_2 - v) - X$ such that $|(X \cup \{z_1, \ldots, z_p\}) \cap V(G_1 \Box C_j)| \leqslant 2$ and is even. Thus, $G_1 \Box C_j - X \cup \{z_1, \ldots, z_p\}$ has a perfect matching M_j for all j $(1 \leqslant j \leqslant l)$. Let M_0' denote the set of edges of M_0 with both ends in X. Then $\bigcup_{i=0}^{l} M_i \cup \{y_1z_1, \ldots, y_pz_p\} - M_0' - \{x_1y_1, \ldots, x_py_p\}$ is a perfect matching of $G_1 \Box G_2 - X$.

From now on, suppose $m \geqslant n \geqslant 2$. Set $|X| = m + n + \varepsilon$. Without loss of generality, we assume $v_1v_2 \in E(G_2)$ satisfying:

$$|X \cap V(G_1^{\{v_1,v_2\}})| = max\{|X \cap V(G_1^{\{v_i,v_j\}})|\,|v_iv_j \in E(G_2), 1 \leqslant i, j \leqslant 2t\}$$

and $|X \cap V(G_1^{v_1})| \geqslant |X \cap V(G_1^{v_2})|$.

Case 1. $|X \cap V(G_1^{\{v_1,v_2\}})| = 1$.

Then, for any $v_iv_j \in E(G_2)$, there are only two possibilities: either $|X \cap V(G_1^{\{v_i,v_j\}})| = 0$ or $|X \cap V(G_1^{\{v_i,v_j\}})| = 1$. Similarly, for any $u_iu_j \in E(G_1)$, $|X \cap V(G_2^{\{u_i,u_j\}})| \leqslant 1$. Otherwise, we can apply induction hypothesis on $(G_1 - \{u_i, u_j\}) \Box G_2$.

Subcase 1.1. m, n are even.

By Lemma 2.5, there exists a perfect matching in $G_1 \Box G_2 - X$.

Subcase 1.2. m and n are odd.

Suppose $|X \cap V(G_1^{v_1})| = 1$ and $|X \cap V(G_1^{v_3})| = 1$ and $v_1v_3 \notin E(G_2)$. Thus $G_1^{v_1} - X$ and $G_1^{v_3} - X$ have perfect matchings M_1 and M_2, respectively, by Theorem 1.1 and the fact that G_1 is m-factor-critical.

Furthermore, G_2 is n-factor-critical with n odd and $n \geqslant 3$, so $G_2 - \{v_1, v_3\}$ is connected and $(n-2)$-factor-critical. By induction hypothesis, $G_1\square(G_2 - \{v_1, v_3\}) - X$ has a perfect matching M_3 as $|X \cap V(G_1\square(G_2 - \{v_1, v_3\}))| = m + n - 1$.

Therefore, $M_1 \cup M_2 \cup M_3$ is a perfect matching of $G_1\square G_2 - X$.

Subcase 1.3. m and n are of different parities.

Assume m is odd and n is even. Suppose $|X \cap V(G_1^{v_1})| = 1 \equiv m \pmod 2$, then $G_1^{v_1} - X$ has a perfect matching M_1 by Theorem 1.1 (1). On the other hand, $G_2 - v_1$ is $(n-1)$-factor-critical with $n-1$ odd. Thus, by induction hypothesis, $G_1\square(G_2 - v_1) - X$ has a perfect matching M_2 as $|X \cap V(G_1\square(G_2 - v_1))| \leqslant m + n$. Hence, $M_1 \cup M_2$ is a perfect matching of $G_1\square G_2 - X$.

Case 2. $2 \leqslant |X \cap V(G_1^{\{v_1,v_2\}})| \leqslant [m+1]_2$ and $n = 2$.

Subcase 2.1. There exists a vertex $v \in V(G_2)$ such that $|X \cap V(G_1^v)| \equiv m \pmod 2$.

In this case, $G_1^v - X$ has a perfect matching M_1 as $|X \cap V(G_1^v)| \leqslant [m+1]_2$. On the other hand, $|X \cap V(G_1\square(G_2 - v))| = [m+3]_2 - |X \cap V(G_1^v)| \leqslant m + 2$ and $|X \cap V(G_1\square(G_2 - v))| \equiv m + 2 \pmod 2$. Since G_2 is bicritical, $G_2 - v$ is 1-factor-critical, and by induction hypothesis, then $G_1\square(G_2 - v) - X$ has a perfect matching M_2. Therefore, $M_1 \cup M_2$ is the desired perfect matching of $G_1\square G_2 - X$.

Subcase 2.2. For any $v \in V(G_2)$, we have $|X \cap V(G_1^v)| \equiv m + 1 \pmod 2$.

Since $|V(G_2)| \geqslant n + 2 = 4$ and $|X| = [m+3]_2$, there exists a vertex $v \in V(G_2)$ such that $1 \leqslant |X \cap V(G_1^v)| \leqslant m - 1$ by the maximality of $|X \cap V(G_1^{\{v_1,v_2\}})| \leqslant [m+1]_2$. Let $Y = V(G_1^v) - X$. Then $|N_{G_1\square(G_2-v)}(Y)| \geqslant 3|Y| \geqslant 3(|V(G_1)| - |X \cap V(G_1^v)|) \geqslant 3(m + 2 - |X \cap V(G_1^v)|)$ as $\delta(G_2) \geqslant 3$. On the other hand, $|X \cap V(G_1\square(G_2 - v))| = [m+3]_2 - |X \cap V(G_1^v)| < 3(m + 2 - |X \cap V(G_1^v)|)$. Hence we can find a vertical edge $e = ww'$ such that $w \in G_1^v - X$ and $w' \in G_1\square(G_2 - v) - X$. Similarly, $G_1^v - (X \cup \{w, w'\})$ has a perfect matching M_1 and by induction hypothesis, $G_1\square(G_2 - v) - (X \cup \{w, w'\})$ has a perfect matching M_2 as $|(X \cup \{w, w'\}) \cap V(G_1\square(G_2 - v))| = [m+3]_2 - |X \cap V(G_1^v)| + 1 \leqslant m + 2$ and $|(X \cup \{w, w'\}) \cap V(G_1\square(G_2 - v))| \equiv m + 2 \pmod 2$. Therefore, $M_1 \cup M_2 \cup \{ww'\}$ is a perfect matching of $G_1\square G_2 - X$.

Case 3. $2 \leqslant |X \cap V(G_1^{\{v_1,v_2\}})| \leqslant [m+1]_2$ and $n \geqslant 3$.

Subcase 3.1. $|X \cap V(G_1^{\{v_1,v_2\}})|$ is odd.

Let $k = |X \cap V(G_1^{\{v_1,v_2\}})|$, $Y = V(G_1^{\{v_1,v_2\}}) - X$ and $H = G_2 - \{v_1, v_2\}$. Then $k \leqslant [m+1]_2 - 1$ and H is connected. Note that each vertex of Y has more than n neighbors in $G_1\square(G_2 - \{v_1, v_2\})$. Hence $|N_{G_1\square H}(Y)| \geqslant n(m + 2 - k)$. Moreover, $|X \cap V(G_1\square H)| = m + n + 1 - k < n(m + 2 - k)$, as $m \geqslant n \geqslant 2$. So, there exists a vertical edge uu' with $u \in V(G_1^{\{v_1,v_2\}}) - X$ and $u' \in V(G_1\square H) - X$. Since $|(X \cup \{u, u'\}) \cap V(G_1^{\{v_1,v_2\}})| \leqslant [m+1]_2$ and is even, then $G_1^{\{v_1,v_2\}} - (X \cup \{u\})$ has a perfect matching M_1.

By induction hypothesis, $G_1\square H - (X \cup \{u'\})$ has a perfect matching M_2 because $|(X \cup \{u, u'\}) \cap V(G_1\square H)| \leqslant m + n + \varepsilon - 2$ and it has the same parity with mn. Therefore, $G_1\square G_2 - X$ has a perfect matching $M_1 \cup M_2 \cup \{uu'\}$.

Subcase 3.2. $|X \cap V(G_1^{\{v_1,v_2\}})|$ is even.

Then $G_1^{\{v_1,v_2\}} - X$ has a perfect matching M_1 and $|X \cap V(G_1 \Box (G_2 - \{v_1, v_2\}))|$ has the same parity with mn. Since $G_2 - \{v_1, v_2\}$ is $(n-2)$-factor-critical by Theorem 1.1, by induction hypothesis, $G_1 \Box (G_2 - \{v_1, v_2\}) - X$ has a perfect matching M_2.

Therefore, $G_1 \Box G_2 - X$ has a perfect matching $M_1 \cup M_2$.

Case 4. $[m+1]_2 + 1 \leqslant |X \cap V(G_1^{\{v_1,v_2\}})| \leqslant m + n + \varepsilon$.

Subcase 4.1. mn is even, say n even.

Then $G_2 - \{v_1, v_2\}$ is $(n-2)$-factor-critical by Theorem 1.1. Set $k = |X \cap V(G_1^{\{v_1,v_2\}})|$. By induction hypothesis, each component of $G_1 \Box (G_2 - \{v_1, v_2\})$ is $(m+n-2)$-factor-critical and thus $G_1 \Box G_2 - X$ has a perfect matching by Lemma 2.3.

Subcase 4.2. mn is odd.

Similarly, we obtain a perfect matching of $G_1 \Box G_2 - X$ by Lemma 2.4. □

Remark 1 The conclusion in Theorem 2.2 is sharp. From Theorem 1.1 (3), there exists an m-factor-critical graph, say G, with minimum degree $m+1$. Then $\delta(G \Box K_2) = m+2$. Assume $d_{G \Box K_2}(u) = m+2$, then the deletion of all neighbors of u in G results in an isolated vertex. Similarly, by sharpness of m-connectivity, we can construct a family of infinite graphs to attain the bound in Theorem 2.1.

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