Sufficient Conditions for n-Matchable Graphs

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Abstract

Let n be a non-negative integer. A graph G is said to be n-**matchable** if the subgraph $G-S$ has a perfect matching for any subset S of $V(G)$ with $|S|=n$. In this paper, we obtain sufficient conditions for different classes of graphs to be n-matchable. Since $2k$-matchable graphs must be k-extendable, we have generalized the results about k-extendable graphs. All results in this paper are sharp.

1 Introduction

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. (Loops and parallel edges are forbidden in this paper.)

For $S \subseteq V(G)$ the induced subgraph of G by S is denoted by $G[S]$. For convenience, we use $G-S$ for the subgraph induced by $V(G)-S$. Denote the number of odd components and components of a graph G by $o(G)$ and $\omega(G)$, respectively. For any vertex x of G, the degree of x is denoted by $d_G(x)$. We define $N(v)=\{u \mid u \in V(G) \text{ and } uv \in E(G)\}$ and $N(S)=\bigcup_{v\in S} N(v)$. Let H be a subgraph of G, we use the notation $N_S(v)=N(v)\cap S$, $N_H(v)=N(v)\cap V(H)$, $d_S(v)=|N_S(v)|$ and $d_H(v)=|N_H(v)|$. Let G and H be two

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graphs. We denote by kH k disjoint copies of H and $G + H$ the **join** of G and H with each vertex of G joining to each vertex of H.

A **matching** in G is a set of edges so that no two of them are adjacent and a **perfect matching** is a matching which covers every vertex of G. A graph G is k-**extendable** if every matching of size k can be extended to a perfect matching. The concept of k-extendable graphs was first introduced by Plummer [9] and since then there has been extensive research done on this topic (e.g., [4], [5] - [12]).

Next, we present the main concept of this paper. Let n be a non-negative integer. A graph G is said to be n-**matchable** where $0 \leq n \leq |V(G)| - 2$ if the subgraph $G - S$ has a perfect matching for any subset S of $V(G)$ with $|S| = n$. The term of n-matchable graphs is first used by Lou in [7] and is also refereed as n-factor-critical graphs by Favaron [2][3] and Yu [12]. This concept is a generalization of notions of factor-critical graphs and bicritical graphs (i.e., cases of $n = 1$ and $n = 2$) in [8]. A characterization of n-matchable graphs is given in [12]. The properties of n-matchable graphs and its relationships with other graph parameters (e.g., degree sum, toughness, binding number, connectivity, etc.) have been discussed in [3], [5] and [7]. It is interesting to notice the fact that if a graph G is $2k$-matchable then it must be k-extendable. Furthermore, if a graph G is $2k$-matchable, then it is still k-extendable by adding any number of edges to it. Thinking of the fact that adding an edge to a k-extendable graph may make it not even 1-extendable (for instance, consider k-extendable bipartite graphs), in this sense $2k$-matchability is a much stronger concept than k-extendability.

In this paper we consider n-matchability of various graphs (such as, claw-free graphs, power graphs, planar graphs, etc.) and obtain sufficient conditions of such graphs to be n-matchable. Therefore we generalize several sufficient conditions of k-extendable graphs to that of $2k$-matchable graphs.

2 Sufficient Conditions for n-Matchable Graphs

We start this section with a few lemmas. The first is a characterization of n-matchable graphs.

LEMMA 2.1 ([12]) Let G be a graph of order p and n an integer such that $0 \leq n \leq p-2$ and $n \equiv p \pmod 2$. Then G is n-matchable if and only if for each subset $S \subseteq V(G)$ with $|S| \geq n$, then $o(G - S) \leq |S| - n$.

The next result shows a relationship between $2n$-matchable graphs and n-extendable graphs.

LEMMA 2.2 ([7]) A graph G of even order is $2n$-matchable if and only if

(a) G is n-extendable; and

(b) for any edge set $D \subseteq E(\bar{G})$, $G \cup D$ is n-extendable.

Applying Euler's formula to planar graphs, we can obtain the following classical result.

LEMMA 2.3 If G is a planar triangle-free graph, then

$$|E(G)| \leq 2|V(G)| - 4$$

With the preparation above, we are ready to prove a sufficient condition for planar graphs to be n-matchable.

THEOREM 2.1 Let G be a 5-connected planar graph of order p. Then G is $(4-\varepsilon)$-matchable, where $\varepsilon = 0$ or 1 and $\varepsilon \equiv p \pmod 2$.
PROOF: Suppose that G is not $(4-\varepsilon)$-matchable. By Lemma 2.1, since G is 5-connected, there exists a subset $S \subseteq V(G)$ with $|S| \geq 5 > 4-\varepsilon$ such that for some $k \geq 1$

$$o(G-S) = |S| - (4-\varepsilon) + 2k \geq 2 \tag{1}$$

We choose S to be as small as possible subject to (1). And let $C_1, C_2, \ldots, C_t$ be the odd components of $G-S$, where $t = |S| - (4-\varepsilon) + 2k$.

We claim that, for each x of S, x is adjacent to at least three of C_1 , C_2, ..., C_t. Otherwise, there is a vertex x in S which is adjacent to at most two of C_1 , C_2, ..., C_t. Let $S' = S - \{x\}$. Then $o(G-S') = |S'| - (4-\varepsilon) + 2q$ for some $q \geq k$ and $|S| > |S'| \geq 4-\varepsilon$, which contradicts to the choice of S or the connectedness of G.

Since G is 5-connected, for each component C of $G-S$ C is adjacent to at least five vertices in S. Now we obtain a bipartite graph H with bipartition (S,Y) by deleting all edges in $G[S]$ and contracting each component of $G-S$ to a vertex and deleting the multiple edges. Then clearly H is planar and triangle free. On the other hand, for each vertex v in S, $d_H(v) \geq 3$, and for each vertex u in Y, $d_H(u) \geq 5$. As G is 5-connected, we have $|S| \geq 5$ and $|Y| \geq |S| - (4-\varepsilon) + 2k \geq 3$. So $|E(H)| \geq \frac{1}{2}(3|S| + 5|Y|)$. Since $|Y| \geq |S| - (4-\varepsilon) + 2$, we can write $|Y| = |S| - (4-\varepsilon) + 2 + m$ for $m \geq 0$. Then

$$|V(H)| = |S| + |Y| = 2|S| - (4-\varepsilon) + 2 + m$$

and

$$\begin{array}{rcl} |E(H)| & \geq & \frac{1}{2}[3|S| + 5(|S| - (4-\varepsilon) + 2 + m)] \\ & = & (4|S| - 2(4-\varepsilon) + 4 + 2m - 4) - \frac{1}{2}(4-\varepsilon) + 5 + \frac{m}{2} \\ & > & 2(|V(H)| - 2) \end{array}$$

This contradicts to Lemma 2.3. □

REMARK 1 Theorem 2.1 implies that a 5-connected planar graph G of even order is 2-extendable, which was proven by Lou [6] and Plummer [10]. Moreover, adding any number of edges to G, the resulting graph (which may not be planar anymore) is still 2-extendable by Lemma 2.2. In fact, any graph of even order having a spanning 5-connected planar subgraph is 2-extendable.

THEOREM 2.2 Let G be a graph of order p and n an integer such that $0 \leq n \leq p-2$ and $n \equiv p \pmod 2$. If G is $(2n+k)$-connected and $K_{1,n+k+2}$-free, then G is n-matchable where $2n+k \geq 1$.
PROOF: Suppose that G is not n-matchable. By Lemma 2.1, there exists a subset $S \subseteq V(G)$ with $|S| \geq 2n+k$ (as G is $(2n+k)$-connected) such that

$$\omega(G-S) \geq o(G-S) \geq |S| - n + 2 \geq 2 \tag{2}$$

Let $C_1, C_2, \ldots, C_t$ be the components of $G-S$, where $t = \omega(G-S)$. Let $e_G(X,Y)$ denote the number of edges with one endvertex in X and the other in Y. Since G is $K_{1,n+k+2}$-free, each vertex u in S is adjacent to at most $n+k+1$ components of $G-S$. Then we have $e_G(X,Y) \leq |S|(n+k+1)$. By the $(2n+k)$-connectedness of G, each C_i is adjacent to at least $2n+k$ vertices in S. Then $e_G(S, G-S) \geq t(2n+k)$. Therefore, $t(2n+k) \leq |S|(n+k+1)$. Recall $|S| \geq 2n+k$ and thus we have

$$\omega(G-S) = t \leq \frac{|S|(n+k+1)}{2n+k} = |S| - \frac{n-1}{2n+k}|S| \leq |S| - n + 1,$$

a contradiction to (2). □

Combining Theorem 2.2 with Lemma 2.2 we have the following corollary which generalizes a result of Sumner [11].

COROLLARY 2.1 If a graph G of even order is $(4n+k)$-connected and $K_{1,2n+k+2}$-free, then G is n-extendable and adding any edge to G the resulting graph is still n-extendable. In other words, every graph of even order that has a $(4n+k)$-connected $K_{1,2n+k+2}$-free spanning subgraph is n-extendable.

The condition of connectivity of Theorem 2.10 is the weakest possible. Let $G_1 = K_{n-1}$, $u_i \notin V(G_1)$, $i = 1, 2, 3, ..., n+k$ and $G_2 = (n+k+1)K_3$, where $V(G_1) \cap V(G_2) = \emptyset$ and $\{u_1, u_2, ..., u_{n+k}\} \cap V(G_2) = \emptyset$. Then we let $G = (G_1 \cup \{u_1, u_2, ..., u_{n+k}\}) + G_2$. Then we can easily see that G is $K_{1,n+k+2}$-free and $\kappa(G) = 2n+k+1$. However, since we have $o(G-(V(G_1) \cup \{u_1, u_2, ..., u_{n+k}\})) = n+k+1 \geq |V(G_1) \cup \{u_1, u_2, ..., u_{n+k}\}| - n = n+k+1$, G is not n-matchable.

Further, $G = (K_n \cup (n+k)K_1) + (n+k+2)K_3$ shows that the upper bound on r for $K_{1,r}$-free graphs in Theorem 2.2 is sharp.

Next we discuss the matchability of power graphs. The rth **power** of a graph G, G^r, is the graph with vertex set $V(G)$ and edge set $\{uv \mid d_G(u,v) \leq r\}$.

THEOREM 2.3 Let G be a graph of order p and n an integer such that $0 \leq n \leq p-2$ and $n \equiv p \pmod 2$.

(a) If G is h-connected and $h > \lfloor \frac{n}{2} \rfloor$, then G^r is n-matchable for $r \geq 2$;

(b) If G is h-connected and $1 \leq h \leq \lfloor \frac{n}{2} \rfloor$, then G^r is n-matchable for $r \geq n-2h+3$.

PROOF: Suppose that G^r is not n-matchable. By Lemma 2.1, there is a subset $S \subseteq V(G)$ with $|S| \geq n$ such that $o(G^r - S) = |S| - n + 2m$ for some $m \geq 1$. Let $S_1 = S - \{v_1, v_2, ..., v_n\}$, where $v_1, v_2, ..., v_n$ are any n vertices in S. Then $o(G^r - S) = |S_1| + 2m$.

(a) For the case of $h > \lfloor \frac{n}{2} \rfloor$, as G is h-connected, each component of $G^r - S$ is adjacent in G to at least h vertices in S. Suppose that no two odd components of $G^r - S$ in G have a common neighbor in S. Then there are at least $(|S_1| + 2m)h$ vertices in S. But S has only $|S| = |S_1| + n < (|S_1| + 2m)h$ vertices, a contradiction. So at least two odd components, say C_1 and C_2, have a common neighbor v in S. Then there is a vertex u in C_1 and a vertex w in C_2 such that $uv \in E(G)$ and $wv \in E(G)$. In G^r, u and w are adjacent. So u and w are in the same component of $G^r - S$, a contradiction to the fact that C_1 and C_2 are different components of $G^r - S$.

(b) For the case of $1 \leq h \leq \lfloor \frac{n}{2} \rfloor$, let $C_1, C_2, ..., C_t$ be the components of $G^r - S$ and let N_i be the set of vertices in S adjacent to vertices of C_i in G. Since G is h-connected, each N_i contains at least h vertices. Furthermore, N_i's are pairwise disjoint. Otherwise, a component C_i contains a vertex u that is distance two from a vertex v in another component C_j. But then u and v would be in the same component of $G^r - S$. Because G is connected, there exists a path P in G from a vertex w_i in N_i to a vertex w_j in $N_j (i \neq j)$. Choose $\bar{P}$ to be such a path with the minimum length among all the path P's. Then $\bar{P}$ is contained in S and none of the internal vertices of $\bar{P}$ is in N_l ($1 \leq l \leq t$). Since $|S| = |S_1| + n$ and $t \geq |S_1| + 2m$, the order of $\bar{P}$ is at most $|S_1| + n - h(|S_1| + 2m) + 2 \leq |S_1| + n - h(|S_1| + 2) + 2 = n - 2h - |S_1|(h-1) + 2 \leq n - 2h + 2$. There is a vertex z_i in C_i and a vertex z_j in C_j adjacent to w_i and w_j, respectively. Then $z_i \bar{P} z_j$ is a path of length at most $n - 2h + 3$. So z_i and z_j are adjacent in G^r, which contradicts to the fact that C_i and C_j are different components of $G^r - S$ again. □

Similar to Remark 1, we can see that Theorem 2.3 implies that for a h-connected graph G of even order its r-power graph G^r is k-extendable where either $k < h$ and $r \geq 2$ or $k \geq h$ and $r \geq 2(k-h)+3$. This result was proven by Holton, Lou and McAvaney in [4].

Our last result is to deal with the n-matchability of total graph $T(G)$.

Total graph $T(G)$ of a graph G is that graph whose vertex set can be put in one-to-one correspondence with the set $V(G) \cup E(G)$ such that two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent or incident. **Subdivision graph** $S(G)$ of a graph G is the graph obtained by replacing all edges of G with paths of length two. Behzad [1] proved that for any graph G, $T(G) = (S(G))^2$.

THEOREM 2.4 Let $T(G)$ be a total graph of order p and n an integer such that $0 \leq n \leq p-2$ and $n \equiv p \pmod 2$. If $T(G)$ is $(n+1)$-connected, then $T(G)$ is n-matchable.

PROOF: Suppose that $T(G)$ is not n-matchable. By Lemma 2.1 and $(n+1)$-connectedness, there exists a *minimal* vertex cut S of $T(G)$ such that $|S| \geq n+1$ and for some $m \geq 1$

$$o(T(G)-S) = |S| - n + 2m \tag{4}$$

We claim that the cut set S contains a subdivision vertex w of $S(G)$. Otherwise, let $P = x_1x_2\ldots x_n$ be a path in G joining two components C_1 and C_2 of $T(G)-S$, where $x_1 \in V(C_1)$ and $x_n \in V(C_2)$. Since $T(G) = (S(G))^2$, then $P' = x_1y_1x_2y_2\ldots x_{n-1}y_{n-1}x_n$ is a path joining x_1 and x_n in $(S(G))^2$, where $y_1, y_2, \ldots, y_{n-1}$ are subdivision vertices of edges $x_1x_2, x_2x_3, \ldots x_{n-1}x_n$. It is easy to see that $y_1y_2\ldots y_{n-1}$ is a path connecting C_1 and C_2 in $(S(G))^2$. Thus, if none of $y_1, y_2, \ldots, y_{n-1}$ is in the cut set S, then there is a path connecting C_1 and C_2 in $T(G) = (S(G))^2$, which contradicts to fact that S is a cut set.

Let w be a subdivision vertex of $S(G)$ in S. Then w is adjacent to at most two components of $T(G)-S$. Set $S_1 = S - \{w\}$, then $o(T(G)-S_1) = |S_1| - n + 2m_1$ for some $m_1 \geq m \geq 1$. If $|S_1| = n$, then it contradicts to the $(n+1)$-connectedness of $T(G)$. If $|S_1| \geq n+1$ and $o(T(G)-S_1) = |S_1| - n + 2m_1$, it contradicts to the minimality of S. $\Box$

REMARK 2 The graphs considered in this paper may have arbitrarily large diameter. We show that adding a new edge to it the resulting graphs are still k-extendable. However, the resulting graphs may not satisfy the original hypotheses in the theorems for those graphs to be k-extendable. So we have found new large families of k-extendable graphs.

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