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Connectivity of k-extendable graphs with large k

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Abstract

Let G be a simple connected graph on $2n$ vertices with perfect matching. For a given positive integer k $(0 \leqslant k \leqslant n-1)$, G is *k-extendable* if any matching of size k in G is contained in a perfect matching of G. It is proved that if G is a k-extendable graph on $2n$ vertices with $k \geqslant n/2$, then either G is bipartite or the connectivity of G is at least $2k$. As a corollary, we show that if G is a maximal k-extendable graph on $2n$ vertices with $n+2 \leqslant 2k+1$, then G is $K_{n,n}$ if $k+1 \leqslant \delta \leqslant n$ and G is K_{2n} if $2k+1 \leqslant \delta \leqslant 2n-1$. Moreover, if G is a minimal k-extendable graph on $2n$ vertices with $n+1 \leqslant 2k+1$ and $k+1 \leqslant \delta \leqslant n$ then the minimum degree of G is $k+1$. We also discuss the relationship between the k-extendable graphs and the Hamiltonian graphs.

Keywords: k-Extendable graph; Minimal k-extendable graph; Maximal k-extendable graph; Minimum degree; Hamiltonian graph

1. Introduction and terminology

All graphs considered in this paper are finite, undirected and simple. For the terminology and notation not defined in this paper, the reader is referred to [4].

Let G and H be two graphs. Let kH denote k disjoint copies of H and $G+H$ denote the union of G and H with each vertex of G joining to every vertex of H.

A graph G is said to be *factor-critical* if $G-v$ has a perfect matching for each $v \in V(G)$. Let G be a graph with a perfect matching. Then G is said to be *k-extendable* for $0 \leqslant k \leqslant (v-2)/2$ if any matching in G of size k is contained in a perfect matching of G. And G is said to be *maximal k-extendable* if G is k-extendable and for each

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$e \in E(\bar{G})$, where $\bar{G}$ is the complement of G, $G \cup \{e\}$ is not k-extendable. And G is said to be *minimal* k*-extendable* if G is k-extendable and for each $e \in E(G)$, $G - e$ is not k-extendable.

The concept of k-extendable graphs was introduced by Plummer [7] in 1980. Since then, extensive researches on this topic have been done (see [1,2,6–10]). In [2], Ananchuen and Caccetta proved the following result about the minimum degree of k-extendable graphs.

Lemma 1 (Ananchuen and Caccetta [2]). *Suppose* $1 \leqslant k \leqslant (v-2)/2$ *and* $|V(G)| = v$. *Then if* G *is* k*-extendable, then either* $k+1 \leqslant \delta \leqslant v/2$ *or* $2k+1 \leqslant \delta \leqslant v-1$.

For each value of δ given in Lemma 1, there exist k-extendable graphs with the minimum degree δ. However, the problem that which value in these ranges is attainable for maximal k-extendable graphs remains open. Plummer [9] proposed the following problem.

Problem 1. Suppose $1 \leqslant k \leqslant (v-2)/2$ and $k+1 \leqslant j \leqslant v/2$ or $2k+1 \leqslant j \leqslant v-1$. Then which k-extendable graphs having minimum degree j are maximal k-extendable?

Motivated by this problem, we study the k-extendable graphs with $k \geqslant v/4$, that is $v/2+1 \leqslant 2k+1$, which means the two intervals for δ in Lemma 1 are separated. We prove that if G is a k-extendable graph with $k \geqslant v/4$, then either G is bipartite or $\kappa(G) \geqslant 2k$. As corollaries, we characterize the maximal k-extendable graphs with $v/2+2 \leqslant 2k+1$ and we show that the minimum degree of a minimal k-extendable graph with $v/2+1 \leqslant 2k+1$ and with $k+1 \leqslant \delta \leqslant v/2$ is $k+1$. Also we prove that a k-extendable graph with $k \geqslant v/4$ is Hamiltonian, which shows the relation between k-extendable graphs and Hamiltonian graphs.

2. Main result

We start this section with a few basic lemmas on k-extendable graphs.

Lemma 2 (Yu [10]). *A graph* G *is* k*-extendable if and only if for any matching* M *of size* r *in* $G (1 \leqslant r \leqslant k)$, $G - V(M)$ *is* $(k-r)$*-extendable.*

Lemma 3 (Yu [10]). *Let* G *be a connected* k*-extendable non-bipartite graph. Then for each edge* $e \in E(\bar{G})$, $G + e$ *is* $(k-1)$*-extendable.*

Lemma 4 (Plummer [7]). *If* G *is* k*-extendable, then* $\kappa(G) \geqslant k+1$.

Lemma 5. *Let* G *be a graph and* $S \subseteq V(G)$. *If the size of a maximum matching of* $G - S$ *is* m, *then the size of a maximum matching of* G *is at most* $m + |S|$.

Proof. Obvious. □

We need the following lemma to prove our main result, this lemma itself may serve as a useful tool in other research on matching theory.

Lemma 6. *Let G be a graph with order $v = 2r + m$. If G has a matching of size r and deleting any vertex from G, the resulting graph still has a matching of size r, then G has a matching of size $r + 1$ unless G has exactly m odd components and no even components and each odd component is factor-critical.*

Proof. Suppose that the maximum matchings of G have size r. Then by Berge's formula on maximum matching, there exists a set $S \subseteq V(G)$ such that $o(G-S)-|S|=m$. If $S \neq \emptyset$, let $v \in S$, $G' = G - v$ and $S' = S \setminus \{v\}$. Then $o(G' - S') - |S'| = o(G - S) - |S| + 1 = m + 1$. So the maximum matching in G' has size at most $(|V(G')| - (o(G' - S') - |S'|))/2 = (2r + m - 1 - (m + 1))/2 = r - 1$, contradicting to the hypothesis that deleting any vertex from G the resulting graph still has a matching of size r. So $S = \emptyset$ and G has exactly m odd components. If G has an even component C, deleting a vertex v from C, $G - v$ has a maximum matching of size less than r since there is a vertex in each of the $m + 1$ odd components which is not covered by the maximum matching and also v is not covered by the maximum matching. Hence, G has no even component. But deleting any vertex v from each odd component C of G, $C - v$ must have a perfect matching, otherwise by counting the number of vertices of G, $G - v$ has no matching of size r. So each component of G is factor-critical. □

Now we give the proof of our main result.

Theorem 7. *If G is a k-extendable graph on v vertices with $k \geqslant v/4$, then either G is bipartite or $\kappa(G) \geqslant 2k$.*

Proof. By contradiction. Suppose that G is a connected k-extendable graph with connectivity at most $2k - 1$ but not bipartite. Let S be a minimum cutset of G and let M be a maximum matching in $G[S]$. Let $T = S \setminus V(M)$ and $r = |M|$. Since $|S| \leqslant 2k - 1$, $|M| \leqslant k - 1$. By Lemmas 2 and 4, $G - V(M)$ is $(k - r + 1)$-connected. Then we have

$$|T| \geqslant k - r + 1 \geqslant 2 \tag{1}$$

and we have $2k - 1 \geqslant 2r + |T| \geqslant k + r + 1$, so

$$r \leqslant k - 2. \tag{2}$$

Claim 1. *For every perfect matching F containing M, $F \cap E(G - S)$ is a maximum matching in $G - S$ and $|F \cap E(G - S)| \leqslant k - 1$.*

Since T is an independent set of G, by (1) and assumption that $|V(G)| \leqslant 4k$,

$$\begin{aligned} |F \cap E(G - S)| &= (|V(G)| - 2|M| - 2|T|)/2 \\ &= |V(G)|/2 - r - |T| \leqslant 2k - (k + 1) = k - 1. \end{aligned}$$

If $F \cap E(G-S)$ is not a maximum matching in $G-S$, then there is a matching F_1 in $G-S$ such that $|F_1| = |F \cap E(G-S)| + 1 \leqslant k$. But by Lemma 5, the size of a maximum matching in $G - V(F_1)$ is at most

$$|V(G-S-V(F_1))| + |M| \leqslant |V(G)|/2 - |F_1| - 1,$$

hence $G - V(F_1)$ does not have perfect matching, this contradicts the k-extendability of G. The proof of Claim 1 is complete. □

By Claim 1 and the fact that T is an independent set of G, we easily prove the following claim.

Claim 2. *The size of every maximum matching in* $G-S$ *is* $|V(G)|/2 - |M| - |T|$.

By (1), there are two distinct vertices x and y in T. By Lemma 3, the graph $H = G + xy$ is $(k-1)$-extendable. By (2), $M_1 = M \cup \{xy\}$ is a matching in H which has size at most $k-1$. Then $H - V(M_1)$ has a perfect matching M^* and M^* matches each vertex of $T \setminus \{x, y\}$ to a vertex in $V(G-S)$. Hence, $M^* \cap E(G-S)$ is a matching of size $|V(G)|/2 - |M| - |T| + 1$ in $G-S$. This contradicts Claim 2. The proof of Theorem 7 is complete. □

Remark 1. The lower bound on connectivity in Theorem 7 is best possible. Let $H_1 = K_{2k}$, $H_2 = K_r$ and $H_3 = K_s$ with $4 \leqslant r + s \leqslant 2k - 2$ and both r and s being positive even integers. Then $G = H_1 + (H_2 \cup H_3)$ is k-extendable but with $\kappa(G) = 2k$. Also the lower bound on k in Theorem 7 is best possible. The hypothesis $k \geqslant v/4$ is equivalent to $v \leqslant 4k$. Let $H_1 = \bar{K}_{k+1}$, $H_2 = K_{k+1}$ and $H_3 = K_{2k}$, where $\bar{K}_{k+1}$ is the complement of K_{k+1}. Then $G = H_1 + (H_2 \cup H_3)$ is a k-extendable graph with $v = 4k + 2$ that is not bipartite but has connectivity $k + 1$.

3. Maximal k-extendable graphs with large k

In this section, we characterize all maximal k-extendable graphs with $v/2+2 \leqslant 2k+1$. Then we show some maximal k-extendable graphs with $2k + 1 \leqslant v/2 + 1$ and with $\delta \geqslant v/2$. Our results partially answer Problem 1.

Lemma 8 (Ananchuen and Caccetta [1]). *If* $G \neq K_v$ *is a maximal* k*-extendable graph on* v *vertices, then*

(a) *if* $v/2 < 2k$, *then* $\delta \leqslant v/2$, *while*
(b) *if* $v/2 \geqslant 2k$, *then* $\delta \leqslant v/2 + 2\lfloor (k-1)/2 \rfloor$.

Lemma 9 (Plummer [8] and Yu [10]). *If* $G = (X, Y) \neq K_{n,n}$ *is a connected* k*-extendable bipartite graph and* $e = xy \in E(\bar{G})$, *where* $x \in X$ *and* $y \in Y$, *then* $G \cup \{e\}$ *is also* k*-extendable.*

Corollary 10. *Let G be a maximal k-extendable graph on v vertices with $v/2 + 2 \leqslant 2k + 1$. Then*

(a) *if $k + 1 \leqslant \delta \leqslant v/2$, then G is $K_{v/2,v/2}$ and hence $\delta = v/2$;*
(b) *if $2k + 1 \leqslant \delta \leqslant v - 1$, then G is K_v and hence $\delta = v - 1$.*

Proof. By Theorem 7, if $k+1 \leqslant \delta \leqslant v/2$, then G is bipartite. Otherwise $\delta(G) \geqslant \kappa(G) \geqslant 2k$. When $v/2+2 \leqslant 2k+1$, $\delta(G) \neq 2k$ by Lemma 1. Hence, $\delta(G) \geqslant 2k+1 \geqslant v/2+2$ and G is non-bipartite. By Lemma 9, we have conclusion (a). By Lemma 8(a), we have conclusion (b). □

Remark 2. Corollary 10 characterizes all maximal k-extendable graphs with $v < 4k$. It shows that the minimum degree of a maximal k-extendable graph G with $v \leqslant 4k-2$ is either $v/2$ or $v-1$. But for the case of $v \geqslant 4k$, we give a family of maximal k-extendable graphs to show that the minimum degree of G can be much more diverse.

Let $G_i = K_{r_i}$, $i = 1,2,\ldots,m$, where each r_i is an odd number and $r_1 + r_2 + \cdots + r_m = 2k - 2 + m$. Let $H_j = K_{s_j}$, $j = 1,2,\ldots,m$, where each s_j is an odd number and $s_1+s_2+\cdots+s_m=2k-2+m$. And let $G=(G_1\cup G_2\cup\cdots\cup G_m)+(H_1\cup H_2\cup\cdots\cup H_m)$. Then it is not too difficult to verify that G is maximal k-extendable but not $(k+1)$-extendable. When we take $m = 2$, by choosing proper r_i and s_i $(i = 1,2)$, we have $\delta(G) = t$ for all even numbers t such that $v/2 \leqslant t \leqslant v/2 + 2\lfloor(k-1)/2\rfloor$. When we take $m = 3$, by choosing proper r_i and s_i $(i = 1,2,3)$, we have $\delta(G) = t$ for all odd numbers t such that $v/2 \leqslant t \leqslant v/2 + \lfloor(2k+1)/3\rfloor - 1$.

4. Minimal *k*-extendable graphs with large *k*

In this section, we show that the minimum degree of a minimal k-extendable graph with $v \leqslant 4k$ and $k+1 \leqslant \delta \leqslant v/2$ is $k+1$. We introduce a result of Lou [6] as a lemma.

Lemma 11 (Lou [6]). *If G is a minimal k-extendable bipartite graph, then $\delta(G) = k + 1$, and furthermore, there are at least $2k + 2$ vertices of degree $k + 1$ in G.*

Corollary 12. *Let G be a minimal k-extendable graph on v vertices with $v/2+1 \leqslant 2k+1$. If $k + 1 \leqslant \delta(G) \leqslant v/2$, then $\delta(G) = k + 1$. Furthermore, there are at least $2k + 2$ vertices of degree $k + 1$ in G.*

Proof. By Theorem 7, if $k + 1 \leqslant \delta(G) \leqslant v/2$, then G is bipartite. By Lemma 11, the result follows. □

Since a k-extendable graph with $k \geqslant v/4$ is rather dense, we make the following conjectures.

Conjecture 1. *Let G be a minimal k-extendable graph on v vertices with $v/2+1 \leqslant 2k+1$. Then $\delta(G) = k + 1$, $2k$ or $2k + 1$.*

In particular, for the case of $v \leqslant 4k-2$, we have the following conjecture.

Conjecture 2. *Let G be a minimal k-extendable graph on v vertices with $v/2+2 \leqslant 2k+1$. If $2k+1 \leqslant \delta \leqslant v-1$, then $\delta(G)=2k+1$.*

5. Hamiltonicity of *k*-extendable graphs with large *k*

In this section, we show that a k-extendable graph is Hamiltonian if k is sufficiently large with respect to its order.

Lemma 13 (Dirac [5]). *If $\delta(G) \geqslant v/2$, then G is Hamiltonian.*

Lemma 14 (Jackson [3]). *Let $G=(X,Y)$ be a connected bipartite graph with $|X|=|Y|=n$. If $\delta(G) \geqslant (n+1)/2$, then G is Hamiltonian.*

Corollary 15. *If G is a k-extendable graph with $k \geqslant v/4$, then G is Hamiltonian.*

Proof. By Theorem 7, if $k+1 \leqslant \delta(G) \leqslant v/2$, $G=(X,Y)$ is bipartite with $|X|=|Y|=v/2 \leqslant 2k$. However, $\delta(G) \geqslant k+1=(2k+2)/2 > (|X|+1)/2$, by Lemma 14, G is Hamiltonian. Otherwise $\delta(G) \geqslant \kappa(G) \geqslant 2k \geqslant v/2$, by Lemma 13, G is Hamiltonian. □

Remark 3. Although we did not find new Hamiltonian graphs in Corollary 15, we did show the relation between k-extendable graphs and Hamiltonian graphs that a k-extendable graph with sufficiently large k with respect to the order $v(G)$ is Hamiltonian. In fact, we suspect that the lower bound on k in Corollary 15 is not best possible. And hence, we give the following conjecture.

Conjecture 3. *If G is a k-extendable graph with $k > (v-2)/6$, then G is Hamiltonian.*

The lower bound on k in Conjecture 3 is best possible. Let $S=\{v_1, v_2, \ldots, v_{2k}\}$ be an independent set and $H=(2k+1)K_2$ with $V(H) \cap S = \emptyset$. Then $G=S+H$ is a k-extendable graph but G is not Hamiltonian as G is not 1-tough. Here $v(G)=6k+2$, that is $k=(v-2)/6$. The above counterexamples also show that a k-extendable graph with arbitrarily large k (but v is also sufficiently large) is not guaranteed to be Hamiltonian.

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