Isolated Toughness and Existence of f-factors *

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Abstract: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *isolated toughness* of G is defined as $I(G) = min\{|S|/i(G-S) \mid S \subseteq V(G), i(G-S) \geq 2\}$ if G is not complete; otherwise, set $I(G) = |V(G)|-1$. Let f and g be two nonnegative integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) \leq b$. The purpose in this paper are to present sufficient conditions in terms of the isolated toughness and the minimum degree for graphs to have f-factors and (g, f)-factors $(g < f)$. If $g(x) \equiv a < b \equiv f(x)$, the conditions can be weakened.

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1 Introduction.

All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For any $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$ and we write $G - S$ for $G[V(G)\backslash S]$. We use $i(G - S)$ to denote the number of isolated vertices of $G - S$. For $S \subseteq V(G)$ and $T \subseteq V(G)$, let $E(S, T) = \{uv \in E(G) \mid u \in S, v \in T\}$ and $e(S, T) = |E(S, T)|$. Other notation and terminology not defined in this paper can be found in [1].

Let g and f be two nonnegative integer-valued functions defined on $V(G)$ and let H be a spanning subgraph of G. We call H a *(g, f)-factor* of G if $g(x) \leq d_H(x) \leq f(x)$ holds for each $x \in V(G)$. Similarly, H is an *f-factor* of G if $g(x) = f(x)$ for each $x \in V(G)$. If $g(x) \equiv a$ and $f(x) \equiv b$ for each $x \in V(G)$, where a, b are positive integers, then a (g, f)-factor is called an *$[a, b]$-factor*.

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For any function $f(x)$ and a vertex subset S, we define $f(S) = \sum_{x \in S} f(x)$.

The well-known necessary and sufficient condition for a graph G to have an f-factor was given by Tutte [9].

Tutte's f-factor Theorem [9]. *A graph G has an f-factor if and only if*

$$f(S) - f(T) + d_{G-S}(T) - o(G - (S \cup T)) \geq 0$$

for any pair of disjoint subsets S and T of $V(G)$, where $o(G-(S \cup T))$ denotes the number of components C of $G - (S \cup T)$ such that $e_G(V(C), T) + \sum_{x \in V(C)} f(x)$ is odd.

For convenience, we denote $\delta(S, T; f) = f(S) - f(T) + d_{G-S}(T) - o(G - (S \cup T))$. So a graph G has an f-factor if and only if $\delta(S, T; f) \geq 0$ for any pair of disjoint S and T. Furthermore, he noticed that $\delta(S, T; f) \equiv \sum_{x \in V(G)} f(x)$ $(mod\ 2)$.

Lovász generalized Tutte's f-factor theorem to (g, f)-factors by minor change in the notion $\delta(S, T; f)$.

Lovász's (g, f)-factor Theorem [7]. *Let G be a graph and g, f be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for any $x \in V(G)$. Then G has a (g, f)-factor if and only if*

$$f(S) - g(T) + d_{G-S}(T) - o(G - (S \cup T)) \geq 0$$

for any pair of disjoint sets $S, T \subseteq V(G)$, where $o(G - (S \cup T))$ denotes the number of components C of $G - (S \cup T)$ such that $g(x) = f(x)$ for any $x \in V(C)$ and $e(V(C), T) + \sum_{x \in V(C)} f(x)$ is odd.

For $g(x) < f(x)$, Heinrich *et al.* [4] simplified Lovász's (g, f)-factor theorem and obtained the following necessary and sufficient condition for the existence of (g, f)-factors.

Lemma 1.1. *(Heinrich et al., [4]) Let g and f be nonnegative integer-valued functions defined on $V(G)$. If either one of the following conditions holds*

(i) $g(x) < f(x)$ for every $x \in V(G)$;

(ii) G is bipartite;

then G has a (g, f)-factor if and only if for any set S of $V(G)$

$$g(T) - d_{G-S}(T) \leq f(S),$$

where $T = \{x \mid x \in V(G) - S,\ d_{G-S}(x) \leq g(x)\}$.

Through the effort of many researchers, there have been many sufficient conditions for the existence of f-factors or (g, f)-factors. For example, the toughness conditions for the existence of some factors are obtained by Katerinis [5] and Chvátal [2]. In particular, Chvátal conjectured that G has k-factors if G is k-tough. This conjecture is confirmed by Enomoto *et al.* [3] and generalized to the following version in [5].

Katerinis' Generalization. *Let G be a graph and $a \le b$ be two positive integers.*

(1) Suppose that $t(G) \ge \frac{(b+a)^2+2(b-a)}{4a}$ when $b \equiv a \ (mod\, 2)$ and $t(G) \ge \frac{(b+a)^2+2(b-a)+1}{4a}$ when $b \not\equiv a \ (mod\, 2)$. If f is an integer-valued function such that $a \le f(x) \le b$ and $\sum_{x\in V(G)} f(x) \equiv 0 \ (mod\, 2)$, then G has an f-factor;

(2) If $t(G) \ge (a-1) + \frac{a}{b}$ and $a|V(G)|$ is even when $a = b$, then G has an $[a,b]$-factor.

The *isolated toughness* was first introduced by Ma and Liu [8] and is motivated from Chvátal's toughness by replacing $c(G-S)$ with $i(G-S)$ in the definition, defined as $I(G) = min\{|S|/i(G-S) \mid S \subseteq V(G),\ i(G-S) \ge 2\}$ if G is not complete; otherwise, set $I(G) = |V(G)| - 1$. Clearly, $I(G) \ge t(G)$ for any graph and $I(G) \le \frac{|V(G)|-\alpha(G)}{\alpha(G)}$, where $\alpha(G)$ is the size of an independent set.

In this paper, we take advantage of the notion of isolated toughness $I(G)$ to obtain several sufficient conditions for the existence of f-factors and (g,f)-factors. The main purpose is to present the following results. Some of them are more general than Katerinis' results.

Theorem 1.1. *Let G be a $K_{1,n}$-free graph and f an integer-valued function on $V(G)$ satisfying $a \le f(x) \le b$ for any $x \in V(G)$ and $\sum_{x\in V(G)} f(x) \equiv 0 \ (mod\, 2)$. If $\delta(G) \ge \frac{(a+b-1)^2+4(b+n-1)}{4(a-n+1)}$ and $I(G) \ge \frac{(a+b-1)^2+4(b+n-1)}{4(a-n+1)}$, where a, b are positive integers satisfying $2 \le n-1 \le a \le b$, then G has an f-factor.*

When the condition $g(x) < f(x)$ for each $x \in V(G)$ is posted, we have the following.

Theorem 1.2. *Let G be a graph and f, g be two nonnegative integer-valued functions with $a \le g(x) < f(x) \le b$. If $\delta(G) \ge \frac{(a+b)^2+2(b-a)+1}{4a}$ and $I(G) \ge \frac{(a+b)^2+2(b-a)+1}{4a}$, where $a \le b$ are two positive integers, then G has a (g,f)-factor.*

For $[a,b]$-factors, the isolated toughness condition in Theorem 1.2 can be weakened. Since its proof is very similar to that of Theorem 1.2, we choose to state the theorem only.

Theorem 1.3. *Let a and b be integers with $2 \le a < b$ and let G be a graph. If $\delta(G) \ge a$ and $I(G) \ge (a-1) + \frac{a}{b}$, then G has an $[a,b]$-factor.*

Let $a = 1 < b$ in Theorem 1.3, then the isolated toughness condition becomes a necessary and sufficient condition for G having $[1,b]$-factors in terms of $I(G)$. This can be derived easily from the criterion of star-factor due to Las Vergnas [6].

Proposition 1.1. *Let G be a graph with $\delta(G) \ge 1$ and $b > 1$ be a positive integer. Then G has a $[1,b]$-factor if and only if $I(G) \ge \frac{1}{b}$*

2 Proof of Theorem 1.1.

A subset I of $V(G)$ is an *independent set* if no two vertices of I are adjacent in G and a subset C of $V(G)$ is a *covering set* if every edge of G has at least one end in C. It is easy to verify that a set $I \subseteq V(G)$ is an independent set of G if and only if $V(G) - I$ is a covering set of G.

To prove the main theorems, we need the following result from Katerinis [5].

Lemma 2.1. *(Katerinis, [5]) Let H be a graph and $S_1, S_2, \cdots, S_{k-1}$ be a partition of $V(H)$ such that $x \in S_j$ if and only if $d_H(x) \leq j$. Then there exist an independent set I and a covering set C of $V(H)$ such that*

$$\sum_{j=1}^{k-1}(k-j)c_j \leq \sum_{j=1}^{k-1} j(k-j)i_j,$$

where $|I \cap S_j| = i_j$ and $|C \cap S_j| = c_j$ for every $j = 1, 2, \cdots, k-1$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1: Suppose, by the contrary, that there exists an integer-valued function f which satisfies all the conditions in the theorem, but G has no f-factors. Then, by Tutte's f-factor Theorem, there exists a pair of disjoint subsets of $V(G)$, say S and T, such that

$$0 > \delta(S, T; f). \tag{2.1}$$

Recall $\delta(S,T;f) = f(S) - f(T) + d_{G-S}(T) - o(G - (S \cup T))$. We choose S and T such that $\delta(S,T;f)$ is the *minimum* and then $|S \cup T|$ is as *large* as possible.

First, we consider the case of $T = \emptyset$. If $S = \emptyset$, then $o(G) = 0$ (since $\sum_{x \in V(G)} f(x)$ is even) and $f(T) - d_{G-S}(T) = 0$. Thus $0 > \delta(S,T;f) = 0$, a contradiction. If $S \neq \emptyset$, then $o(G - S) \leq (n-1)|S|$ since G is $K_{1,n}$-free. Then $(n-1)|S| \geq o(G - S) > f(S) \geq a|S|$ by (2.1) and thus $a < n - 1$, which contradicts to the condition given in the theorem.

So, we may assume that $T \neq \emptyset$. Next we prove the following two claims.

Claim 1. $i(G - (S \cup T)) = 0$.

If $i(G-(S\cup T)) \neq 0$, then there exists an isolated vertex, say v, in $G-(S\cup T)$. If $e_G(v,T) > f(v)$, then set $S' = S \cup \{v\}$ and we have

$$\begin{aligned}
\delta(S',T;f) &= f(S') + d_{G-S'}(T) - f(T) - o(G - (S' \cup T)) \\
&\leq f(S) + f(v) + d_{G-S}(T) - e_G(v,T) - f(T) - (o(G - (S \cup T)) - 1) \\
&= \delta(S,T;f) + f(v) + 1 - e_G(v,T) \\
&\leq \delta(S,T;f),
\end{aligned}$$

which contradicts to the maximum of $|S \cup T|$ with respect to the minimum of $\delta(S,T;f)$. If $e_G(v,T) \leq f(v)$, then set $T' = T \cup \{v\}$ and we have

$$\delta(S,T';f) = f(S) + d_{G-S}(T') - f(T') - o(G - (S \cup T'))$$

$$
\begin{aligned}
&\le f(S) + d_{G-S}(T) + e_G(v,T) - f(T) - f(v) - (o(G-(S\cup T)) - 1)\\
&= \delta(S,T;f) + e_G(v,T) - f(v) + 1\\
&\le \delta(S,T;f) + 1.
\end{aligned}
$$

Since $\delta(S,T';f) \equiv \sum_{x\in V(G)} f(x) \equiv \delta(S,T;f)\ (mod\, 2)$, we have $\delta(S,T';f) \le \delta(S,T;f)$. Again this is a contradiction to the maximum of $|S\cup T|$ with respect to the minimum of $\delta(S,T;f)$. Therefore, $i(G-(S\cup T)) = 0$.

Claim 2. $d_{G-S}(x) \le b+n-1$ for any $x \in T$.

For any $x \in T$, let $T' = T - \{x\}$. By the minimum of $\delta(S,T;f)$, we have $\delta(S,T';f) \ge \delta(S,T;f)$. Since G is a $K_{1,n}$-free graph, x is adjacent to at most $n-1$ components of $G-(S\cup T)$ or $o(G-(S\cup T')) \ge o(G-(S\cup T)) - (n-1)$. Therefore

$$
\begin{aligned}
\delta(S,T;f) &\le \delta(S,T';f) = f(S) - f(T') + d_{G-S}(T') - o(G-(S\cup T'))\\
&\le f(S) - f(T) + f(x) + d_{G-S}(T) - d_{G-S}(x) - (o(G-(S\cup T)) - (n-1))\\
&= \delta(S,T;f) + f(x) - d_{G-S}(x) + n - 1.
\end{aligned}
$$

Thus $d_{G-S}(x) \le b+n-1$ as $f(x) \le b$.

Let $T^j = \{x \mid x \in T,\ d_{G-S}(x) = j\}$, $t_j = |T^j|$ for every $j = 0,1,2,\cdots,b+n-1$ and $H = G[T^1 \cup T^2 \cup \cdots \cup T^{b+n-1}]$. Then T^0 is the set of the isolated vertices and $\{T^j \mid j = 1,2,\cdots,b+n-1\}$ is a vertex partition of H. Applying Lemma 2.1 with $k = b+n$, then there exist an independent set I and a covering C of $V(H)$ such that

$$
\sum_{j=1}^{b+n-1} (b+n-1-j)c_j \le \sum_{j=1}^{b+n-1} j(b+n-1-j)i_j, \tag{2.2}
$$

where $|I \cap T^j| = i_j$ and $|C \cap T^j| = c_j$ for every $j = 1,2,\cdots,b+n-1$. Clearly, Lemma 2.1 holds for any independent set $I' \supseteq I$ and the cover set C. So, without loss of generality, we may assume that I is a maximal independent set.

Set $W = G-(S\cup T)$ and $U = S \cup C \cup (N_G(I) \cap V(W))$. Then

$$
|U| \le |S| + \sum_{j=1}^{b+n-1} j i_j, \tag{2.3}
$$

$$
i(G-U) \ge \sum_{j=1}^{b+n-1} i_j + t_0. \tag{2.4}
$$

Case 1. $i(G-U) \ge 2$.

Since $i(G-U) \ge 2$, by the definition of $I(G)$,

$$
|U| \ge i(G-U)I(G). \tag{2.5}
$$

Combining (2.3), (2.4) and (2.5), we have

$$|S| \geq \sum_{j=1}^{b+n-1}(I(G)-j)i_j + I(G)t_0. \tag{2.6}$$

Since $a \leq f(x) \leq b$ for each $x \in V(G)$, so $o(G-(S\cup T)) > f(S)-f(T)+d_{G-S}(T) \geq a|S|-b|T|+d_{G-S}(T)$. On the other hand, since G is a $K_{1,n}$-free graph, we have $o(G-(S\cup T)) \leq (n-1)(|S|+|T|)$. Thus $(n-1)(|S|+|T|) > a|S|-b|T|+d_{G-S}(T)$ and this implies that

$$(b+n-1)|T| - d_{G-S}(T) > (a-n+1)|S|. \tag{2.7}$$

However, $(b+n-1)|T|-d_{G-S}(T) = \sum_{j=0}^{b+n-1}(b+n-1-j)t_j \leq \sum_{j=1}^{b+n-1}(b+n-1-j)i_j + \sum_{j=1}^{b+n-1}(b+n-1-j)c_j + (b+n-1)t_0$, since $t_j \leq c_j + i_j$ in T. Thus

$$\sum_{j=1}^{b+n-1}(b+n-1-j)i_j + \sum_{j=1}^{b+n-1}(b+n-1-j)c_j + (b+n-1)t_0 > (a-n+1)|S|. \tag{2.8}$$

Combining (2.8) and (2.6), we have

$$\sum_{j=1}^{b+n-1}(b+n-1-j)c_j > \sum_{j=1}^{b+n-1}[(a-n+1)(I(G)-j)-(b+n-1-j)]i_j + [(a-n+1)I(G)-(b+n-1)]t_0. \tag{2.9}$$

Notice that $(a-n+1)I(G)-(b+n-1) > 0$ since $I(G) \geq \frac{(a+b-1)^2+4(b+n-1)}{4(a-n+1)}$. Therefore, (2.9) implies that

$$\sum_{j=1}^{b+n-1}(b+n-1-j)c_j > \sum_{j=1}^{b+n-1}[(a-n+1)(I(G)-j)-(b+n-1-j)]i_j. \tag{2.10}$$

By (2.10) and (2.2), we have

$$\sum_{j=1}^{b+n-1} j(b+n-1-j)i_j > \sum_{j=1}^{b+n-1}[(a-n+1)(I(G)-j)-(b+n-1-j)]i_j.$$

Hence there exists some $j \in \{1, 2, \cdots, b+n-1\}$ such that $j(b+n-1-j) > (a-n+1)(I(G)-j)-(b+n-1-j)$, that is, $j(b+a-j-1) > (a-n+1)I(G)-b-n+1$.

Let $h(j) = j(b+a-1-j)$. The maximum value of $h(j)$ is $\frac{(a+b-1)^2}{4}$ when $j = \frac{a+b-1}{2} \leq b+n-1$. But $(a-n+1)I(G)-b-n+1 \geq \frac{(a+b-1)^2}{4} \geq j(b+a-j-1)$ for any $j \in \{1, 2, \cdots, b+n-1\}$ since $I(G) \geq \frac{(a+b-1)^2+4(b+n-1)}{4(a-n+1)}$, a contradiction.

Case 2. $i(G-U) = 0$.

By (2.4), we have $\sum_{j=1}^{b+n-1} i_j + t_0 \leq 0$ or $t_0 = i_j = 0$ for all $j = 1, 2, \cdots, b + n - 1$. Since I is an maximal independent set, we have $T = \emptyset$, a contradiction to our assumption that $T \neq \emptyset$.

Case 3. $i(G - U) = 1$.

Then, by (2.4), we have $\sum_{j=1}^{b+n-1} i_j + t_0 \leq 1$.

If $t_0 = i_j = 0$ for all $j = 1, 2, \cdots, b + n - 1$, it is exactly Case 2.

If $t_0 = 1$ and $i_j = 0$ for all $j \in \{1, \cdots, b+n-1\}$, then T is an isolated vertex, say v. Therefore, by $o(G - (S \cup T)) \leq (n-1)(|S| + |T|) = (n-1)(|S| + 1)$ and $o(G - (S \cup T)) > f(S) - f(T) + d_{G-S}(T) \geq a|S| - b$, it yields

$$(a - n + 1)|S| < b + n - 1. \tag{2.11}$$

On the other hand,

$$\delta(G) \leq d_G(v) = e(v, S) \leq |S|. \tag{2.12}$$

Thus, by (2.11) and (2.12), we have $\delta(G)(a - n + 1) < b + n - 1$. But this is impossible because $\delta(G)(a - n + 1) - (b + n - 1) \geq \frac{(b+a-1)^2}{4} > 0$ since $\delta(G) \geq \frac{(a+b-1)^2+4(b+n-1)}{4(a-n+1)}$.

If there exists some $j_0 \in \{1, 2, \cdots, b+n-1\}$ such that $i_{j_0} = 1$ and $i_j = t_0 = 0$ for all $j \in \{1, 2, \cdots, b + n - 1\} \backslash j_0$, then the maximality of I implies that H is a complete graph. Let $I = \{u\}$ for some vertex $u \in V(H)$. Then $d_G(u) \leq |S| + j_0$ and so $|S| \geq \delta(G) - j_0$. By (2.8), we have

$$\sum_{j=1}^{b+n-1} (b + n - 1 - j)c_j > (a - n + 1)(\delta(G) - j_0) - (b + n - 1 - j_0). \tag{2.13}$$

Combining (2.2) and (2.13), we get $j_0(b+n-1-j_0) > (a-n+1)(\delta(G)-j_0) - (b+n-1-j_0)$. The maximum value of $j_0(b+n-1-j_0) + (a-n+1)j_0 - j_0 = j_0(b + a - j_0 - 1)$ is $\frac{(a+b-1)^2}{4}$, but $(a - n + 1)\delta(G) - (b + n - 1) \geq \frac{(a+b-1)^2}{4}$ since $\delta(G) \geq \frac{(a+b-1)^2+4(b+n-1)}{4(a-n+1)}$, a contradiction again.

In all the cases, we derive a contradiction and thus complete the proof. ∎

3 Proof of Theorem 1.2.

In this section, we provide a proof for Theorem 1.2.

Proof of Theorem 1.2:

Suppose that there exist two functions g and f which satisfy the conditions of the theorem but G has no (g, f)-factors. Then, by Lemma 1.1, there exists a vertex set $S \subset V(G)$ such that

$$g(T) - d_{G-S}(T) > f(S), \tag{3.1}$$

where $T = \{x \mid x \in V(G) - S,\ d_{G-S}(x) \leq g(x)\}$.

Choose T such that T is minimal subject to (3.1). Suppose that there exists $x \in T$ such that $d_G(x) = g(x)$. Then the sets S and $T - \{x\}$ satisfy (3.1), which contradicts to the choice of T. Hence we have $d_G(x) \leq g(x) - 1$ for all $x \in T$.

We assume that $S \neq \emptyset$. Otherwise, since $\delta(G) \geq \frac{(a+b)^2+2(b-a)+1}{4a} > b \geq g(x)$ for every $x \in V(G)$, thus $T = \emptyset$ and (3.1) does not hold. Without loss of generality, we use $T = \{x \mid x \in V(G) - S,\ d_{G-S}(x) \leq b - 1\}$ instead of $T = \{x \mid x \in V(G) - S,\ d_{G-S}(x) \leq g(x) - 1\}$ since $g(x) \leq b$ for every $x \in V(G)$.

For each $0 \leq i \leq b - 1$, let $T^i = \{x \mid x \in T,\ d_{G-S}(x) = i\}$ and $t_i = |T^i|$ (we allow $T^i = \emptyset$ for some i), then T^0 is the set of the isolated vertices. Let $H = G[T^1 \cup T^2 \cup \cdots \cup T^{b-1}]$, then $d_H(x) \leq i$ for each $x \in T^i$ and $\{T^i \mid i = 1, 2, \cdots b-1\}$ is a vertex partition of H. By Lemma 2.1, there exist an independent set I and a covering set C of $V(H)$ such that

$$\sum_{j=1}^{b-1}(b-j)c_j \leq \sum_{j=1}^{b-1} j(b-j)i_j, \tag{3.2}$$

where $|I \cap T^j| = i_j$ and $|C \cap T^j| = c_j$ for every $j = 1, 2, \cdots, b - 1$.

Without loss of generality, we may choose I to be a maximal independent set of H. Set $W = G - (S \cup T)$ and $U = S \cup C \cup (N_{G-S}(I) \cap V(W))$. Then

$$|U| \leq |S| + \sum_{j=1}^{b-1} j i_j, \tag{3.3}$$

$$i(G-U) \geq t_0 + \sum_{j=1}^{b-1} i_j. \tag{3.4}$$

Case 1. $i(G - U) \geq 2$.

By the definition of $I(G)$,

$$|U| \geq i(G-U)I(G). \tag{3.5}$$

Combining (3.3), (3.4) and (3.5), we have

$$|S| \geq \sum_{j=1}^{b-1}(I(G)-j)i_j + I(G)t_0. \tag{3.6}$$

On the other hand, since $g(x) \leq b$ for every $x \in V(G)$, $g(T) - d_{G-S}(T) \leq b|T| - d_{G-S}(T) = \sum_{j=0}^{b-1}(b-j)t_j \leq \sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j + bt_0$. Since $f(x) \geq a$ for every $x \in V(G)$, we have $f(S) \geq a|S|$. From (3.1) and (3.6), we obtain

$$\sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j > a|S| \geq a\sum_{j=1}^{b-1}(I(G)-j)i_j + (aI(G)-b)t_0 \geq a\sum_{j=1}^{b-1}(I(G)-j)i_j,$$

this implies that

$$\sum_{j=1}^{b-1}(b-j)c_j > \sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j. \tag{3.7}$$

By (3.7) and (3.2), we have

$$\sum_{j=1}^{b-1} j(b-j)i_j > \sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j.$$

Hence, there exists some $j \in \{1,2,\cdots,b-1\}$ such that $j(b-j) > aI(G)-aj-b+j$. But $j(b-j)+aj-j = -j^2+(a+b-1)j \le \frac{(a+b-1)^2}{4}$ and $aI(G)-b \ge \frac{(b+a)^2+2(b-a)+1}{4} - b = \frac{(b+a-1)^2}{4}$ due to $I(G) \ge \frac{(b+a)^2+2(b-a)+1}{4a}$, a contradiction.

Case 2. $i(G-U)=0$.

By (3.4), we have $0 \ge t_0 + \sum_{j=1}^{b-1} i_j$ or $t_0 = i_j = 0$ for all $j = 1,2,\cdots b-1$. Since I is maximal, it yields $T=\emptyset$. Hence $g|T| - d_{G-S}(T) = 0 > g(S) > 0$, a contradiction.

Case 3. $i(G-U)=1$ or $1 \ge t_0 + \sum_{j=1}^{b-1} i_j$ from (3.4).

If $t_0 = i_j = 0$ for all $j=1,2,\cdots,b-1$, it is exactly Case 2.

If $t_0 = 1$, then for all $j=1,2,\cdots,b-1$, $i_j = 0$ and T is an isolated vertex. Let $T=\{v\}$. Since $a \le g(x) < f(x) \le b$ for every $x \in V(G)$, we have $g(T) - d_{G-S}(T) = g(v) \le b$ and $f(S) \ge a|S| \ge ad_G(v) \ge a\delta(G) \ge \frac{(b+a)^2+2(b-a)+1}{4} \ge b$, a contradiction to (3.1).

Suppose there exists some $j_0 \in \{1,2,\cdots,b-1\}$ such that $i_{j_0} = 1$ and $i_j = 0$ for all $j \in \{1,2,\cdots,b-1\} - \{j_0\}$. Since I is maximal, then H is a complete graph. From (3.3), we have $|U| \le |S| + j_0$. On the other hand, $|U| \ge |S| + d_{G-S}(v) \ge \delta(G)$ and it yields

$$|S| \ge |U| - j_0 \ge \delta(G) - j_0. \tag{3.8}$$

On the other hand, $g(T) - d_{G-S}(T) \le b|T| - d_{G-S}(T) = \sum_{j=1}^{b-1}(b-j)t_j \le \sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j = (b-j_0) + \sum_{j=1}^{b-1}(b-j)c_j$. Since $f(x) \ge a$ for every $x \in V(G)$, by (3.8), we have $f(S) \ge a|S| \ge a(\delta(G)-j_0)$. These inequalities imply

$$\sum_{j=1}^{b-1}(b-j)c_j + (b-j_0) > a(\delta(G)-j_0), \tag{3.9}$$

and by (3.2),

$$\sum_{j=1}^{b-1}(b-j)c_j \le j_0(b-j_0). \tag{3.10}$$

Thus, from (3.9) and (3.10), we have

$$j_0(b-j_0) > a(\delta(G)-j_0) - (b-j_0) \ge a(\frac{(a+b)^2+2(b-a)+1}{4a} - j_0) - (b-j_0)$$

or

$$\frac{(a+b)^2+2(b-a)+1}{4} - b < -j_0^2 + (a+b-1)j_0. \tag{3.11}$$

However, for any j_0 ($1 \le j_0 \le b-1$), it is not hard to see that $(a+b)^2+2(b-a)+1-4b \ge (a+b-1)^2-(2j_0-(a+b-1))^2$ or equivalently $\frac{(a+b)^2+2(b-a)+1}{4} - b \ge \frac{(a+b-1)^2}{4} - (j_0 - \frac{a+b-1}{2})^2 = -j_0^2 + (a+b-1)j_0$, a contradiction to (3.11).

The proof is complete. ■

Although all the graphs considered are simple, the theorems in this paper can be extended to graphs with multiple edges as well (but without loops). To see this, one needs only to notice that a graph with multiple edges has the same isolated toughness as its underlying graph.

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