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On Tutte polynomial uniqueness of twisted wheels

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Abstract

A graph G is called T-*unique* if any other graph having the same Tutte polynomial as G is isomorphic to G. Recently, there has been much interest in determining T-unique graphs and matroids. For example, de Mier and Noy [A. de Mier, M. Noy, On graphs determined by their Tutte polynomials, Graphs Combin. 20 (2004) 105–119; A. de Mier, M. Noy, Tutte uniqueness of line graphs, Discrete Math. 301 (2005) 57–65] showed that wheels, ladders, Möbius ladders, square of cycles, hypercubes, and certain class of line graphs are all T-unique. In this paper, we prove that the twisted wheels are also T-unique.

Keywords: Tutte polynomial; T-unique; Twisted wheels

1. Introduction

Let G be a graph with vertex set V and edge set E. We assume that G has no isolated vertices, but loops and multiple edges are allowed. The *rank* of a subset S of E is the number of edges in the spanning forest of the subgraph induced by S in G, i.e. $r(S) = |V| - k(G|S)$, where $k(G|S)$ denotes the number of components of the spanning subgraph induced by S in G. The *Tutte polynomial* of G is defined as

$$T(G; x, y) = \sum_{S \subseteq E} (x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)}.$$

The Tutte polynomial was introduced in 1954 by Tutte [12] as a generalization of the chromatic polynomial and as a tool to attack the four-color conjecture.

Two graphs G_1 and G_2 are called *Tutte polynomial equivalent*, or T-*equivalent* for short, if $T(G_1; x, y) = T(G_2; x, y)$. For a graph G, if any graph H having the same Tutte polynomial with G implies that $H \cong G$, then G is called T-*unique* [9]. Clearly, not every graph is T-unique, for instance, all trees of n vertices have the same Tutte polynomial. Furthermore, Bollobás, Pebody and Riordan [3] constructed non-isomorphic graphs of arbitrarily high connectivity with the same Tutte polynomial.

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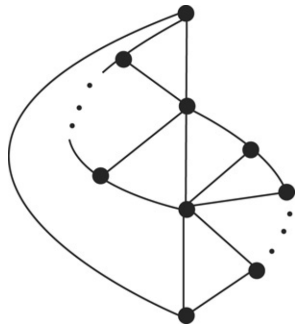


Fig. 1. The twisted wheel.

The coefficients of the Tutte polynomial of a graph contain a lot of information about the graph, such as the graphic parameters shown in the next two theorems. We will use these results frequently in our later proofs.

Theorem 1.1. *(de Mier and Noy [9, Theorem 2.4])*

Let $G = (V, E)$ be a 2-connected graph, then the following graphic parameters of G are determined by its Tutte polynomial:

(i) *The number of vertices and the number of edges;*
(ii) *For every k, the number of edges with multiplicity k. In particular, whether G is a simple graph or not;*
(iii) *The number of shortest cycles;*
(iv) *The edge-connectivity $\lambda(G)$;*
(v) *If G is simple, the number of cliques of each size. In particular, the clique-number $\omega(G)$;*
(vi) *If G is simple, the number of cycles of lengths three, four and five. For the cycles of length four, it is also possible to know how many of them have exactly one chord.*

Let $n(G) = |E(G)| - r(G)$. The following theorem is a well-known result, here formulated for graphic matroids (see [9, Theorem 2.2]). A slightly more general result for matroids can be found, for example, in [5, Example 6.2.17].

Theorem 1.2. *Suppose that $T(G; x, y) = \sum b_{ij} x^i y^j$. If both of $r(G)$ and $n(G)$ are positive, then the number of blocks of G is $\min\{i \mid b_{i0} \neq 0\}$. Otherwise, G has $|E(G)|$ blocks. In particular, if G is 2-connected and H is T-equivalent to G, then H is also 2-connected.*

A *bond* of a graph is a minimal edge-cut. In [6], it is proved that for a 2-connected graph G with $\lambda(G) \geq 3$, the number of the minimum bonds is determined by its flow polynomial. As the flow polynomial of a graph is an evaluation of its Tutte polynomial (see, for example, [2]), the number of minimum bonds of a graph can also be determined by its Tutte polynomial. We will use this fact in Section 3.

Based on Theorems 1.1 and 1.2, several classes of graphs have been proved to be T-unique. de Mier and Noy [9] proved that wheels W_n, square of cycles C_n^2, complete multipartite graphs $K_{p_1,p_2,\cdots,p_r}$, ladders L_n, Möbius ladders M_n and hypercubes Q_n are T-unique. In [7], it is proved that generalized Peterson graph $P(m, 2)$ is T-unique. de Mier and Noy [10] considered T-uniqueness of line graphs and proved that the line graphs of complete graphs K_n, complete bipartite graphs $K_{p,q}$ and regular complete t-partite graphs $K(p, t)$ $(t \geq 2)$ are T-unique.

Márquez, de Mier, Noy and Revuelta [8] proved that the locally grid graphs are T-unique. Bonin and de Mier [4] studied T-uniqueness of certain class of matroids. Here, we will continue the research of T-uniqueness of graphs. We study T-uniqueness of twisted wheels which are obtained by adding an edge to two fans sharing a common edge (see Fig. 1). The class of twisted wheels is one of two classes of graphs with exactly two non-essential edges in 3-connected graphs [13,11] (an edge in a 3-connected graph G is *non-essential* if either $G \setminus e$ or G/e is both simple and 3-connected). Moreover, the twisted wheels form a subclass of another class of graphs, *accordion graphs*, defined by Benashski, Martin, Moore and Traldi [1]. In this paper, we will show that all twisted wheels with at least two internal spokes in each fan are T-unique. The formal statement and the proof will be given in Section 3.

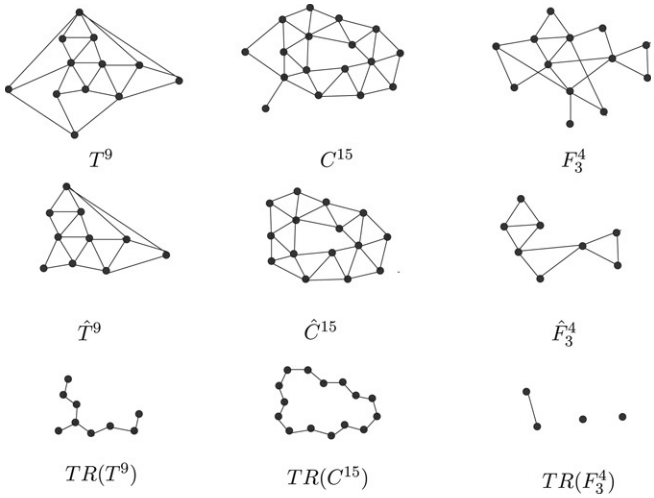


Fig. 2. Examples of triangle-induced subgraphs and triangle-graphs.

2. The triangle-graph of a graph

From Theorem 1.1(ii), we see that any graph which is T-equivalent to a simple graph is also simple. Although the Tutte polynomials are defined on all graphs, in this paper we restrict our attention to T-uniqueness of simple graphs only. From now on, all graphs considered are simple.

Definition 2.1. For a graph G, the subgraph induced by all edges contained in a triangle of G is called the *triangle-induced subgraph* of G, denoted by $\hat{G}$.

Similar to the concept of line graphs, we introduce the following term.

Definition 2.2. For a graph G, define a new graph $TR(G)$ associated with G as follows. Each vertex of $TR(G)$ corresponds to a triangle in G and two vertices are adjacent in $TR(G)$ if and only if the corresponding triangles share an edge in G.

Clearly, the graph $TR(G)$ is well-defined and simple. We refer to $TR(G)$ as the *triangle-graph* of G. In the following, we use C_4^+ to denote a cycle of length four with exactly one chord. The graph consisting of q triangles sharing a common edge is denoted by $K_{q,2}^+$. Clearly, the triangle-graph of $K_{q,2}^+$ is a complete graph K_q. Moreover, the number of vertices in $TR(G)$ is equal to the number of triangles in G and the number of edges in $TR(G)$ is equal to the number of C_4^+ in G.

Definition 2.3. Let $TR(G)$ be the triangle-graph of G.

(i) If $TR(G)$ is a tree of n vertices, then the graph G is called a *triangular n-tree graph*. Denote the set of such graphs by $\mathit{\Gamma}^n$. Furthermore, if $TR(G)$ is a path of order n, then G is called a *triangular n-path graph* and the set of such graphs is denoted by $\wp^n$.
(ii) If $TR(G)$ is a cycle of length n, then G is called a *triangular n-cycle graph*, and we denote the set of such graphs by $\mathcal{C}^n$.
(iii) If $TR(G)$ is a forest of n vertices with r components, then G is called a *triangular n-forest with r components graph*, and we denote the set of such graphs by $\mathcal{F}_r^n$.

We also use T^n to denote a graph in $\mathit{\Gamma}^n$, P^n a graph in $\wp^n$, C^n a graph in $\mathcal{C}^n$ and F_r^n a graph in $\mathcal{F}_r^n$, respectively. Some examples are shown in Fig. 2.

Proposition 2.4. (i) *If $T^n \in \mathit{\Gamma}^n$, then $|V(\hat{T}^n)| \le n+2$ and $|E(\hat{T}^n)| = 2n+1$;*
(ii) *If $C^n \in \mathcal{C}^n$, $n \ge 4$, then $|V(\hat{C}^n)| \le n+1$ and $|E(\hat{C}^n)| = 2n$.*

Proof. (i) Since there are n triangles in $\hat{T}^n$, and at least $2(n-1)$ vertices belong to more than one triangle, we conclude that $|V(\hat{T}^n)| \le 3n - 2(n-1) = n+2$.

Now we show that $|E(\hat{T}^n)| = 2n+1$. Recall that the number of vertices in $TR(G)$ is equal to the number of triangles in G and the number of edges in $TR(G)$ is equal to the number of C_4^+'s in G. By Definition 2.3(i), there are n triangles and $n-1$ C_4^+'s in T^n. Since T^n is a triangular n-tree graph, it contains no $K_{3,2}^+$ as a subgraph. Thus there are $3n-(n-1) = 2n+1$ edges contained in triangles. Hence $|E(\hat{T}^n)| = 2n+1$.

(ii) Let $e = uv$ be an edge of $\hat{C}^n$ which is contained in exactly one triangle. Note that such an edge does exist. Then both u and v are still contained in some triangles of $\hat{C}^n - e$. Moreover, $\hat{C}^n - e$ is a graph in the class Γ^{n-1}. By (i), $\hat{C}^n - e$ contains at most $n+1$ vertices and thus $|V(\hat{C}^n)| = |\hat{C}^n - e| \le n+1$. On the other hand, $|E(\hat{C}^n)| = |E(\hat{T}^{n-1})| + 1 = 2(n-1)+1+1 = 2n$, as required. □

Definition 2.5. Suppose that G is a connected graph in Γ^n such that $V(G) = V(\hat{G})$, $E(G) = E(\hat{G})$ and $|V(\hat{G})| = n+2$. Then G is called a *maximal triangular n-tree graph*, denoted by $T^n_{\max}$. The set of all such graphs is denoted by $\Gamma^n_{\max}$.

Similarly, we denote the *maximal triangular n-path graph* by $P^n_{\max}$ and the set of such graphs by $\wp^n_{\max}$; the *maximal triangular n-cycle graph* by $C^n_{\max}$ and the set of such graphs by $\mathcal{C}^n_{\max}$. For example, in Fig. 2, neither T^9 nor $\hat{T}^9$ is a maximal triangular 9-tree graph; $\hat{C}^{15}$ is a maximal triangular 15-cycle graph but C^{15} is not.

In the proof of Proposition 2.4, the vertices of $T^n_{\max}$ are counted repeatedly only in the case of two triangles sharing an edge. Thus, a maximal triangle-induced n-tree graph $T^n_{\max}$ has at least two vertices of degree 2. In particular, a maximal triangle-induced n-path graph $P^n_{\max}$ has exactly two vertices of degree 2.

It is clear that in a tree, the number of paths of length two (or 2-paths) is equal to the number of edges of its line graph. Let $L(G)$ denote the line graph of G. It is well-known that $|E(L(G))| = \sum_{v \in V} \binom{d(v)}{2}$. Next, we will count the number of 2-paths in a tree.

Lemma 2.6. *Let T be a tree with n vertices and maximum degree $\Delta(T)$. Let m_i be the number of vertices of degree i in T. Then the number of 2-paths is $n-2+\sum_{i=3}^{\Delta(T)} m_i \binom{i-1}{2}$, and hence T has $n-2$ 2-paths if and only if T is an n-path.*

Proof. Since $|E(T)| = n-1 = \sum_{i=1}^{\Delta(T)} i \cdot m_i$ and $|V(T)| = n = \sum_{i=1}^{\Delta(T)} m_i$, the number of 2-paths is

$$\sum_{i=1}^{\Delta(T)} m_i \binom{i}{2} = \sum_{i=1}^{\Delta(T)} m_i \binom{i-1}{2} + \sum_{i=1}^{\Delta(T)} i \cdot m_i - \sum_{i=1}^{\Delta(T)} m_i = \sum_{i=3}^{\Delta(T)} m_i \binom{i-1}{2} + n - 2.$$

The result then follows immediately. □

3. Main result

In this section, we will state and prove our main result. A *twisted wheels* W_{k_1,k_2} is a graph obtained from K_4 with vertex set $\{v_1, v_2, v_3, v_4\}$ by subdividing the edges v_1v_2 and v_3v_4 with k_1-1 and k_2-1 vertices, and then joining each of the new vertices on v_1v_2 and v_3v_4 to v_3 and v_1, respectively (see Fig. 3 where v_1, v_2, v_3, v_4 have been relabeled as v_{1,k_1}, $v_{1,0}$, v_{2,k_2}, and $v_{2,0}$, respectively). Clearly, W_{k_1,k_2} has k_1+k_2 triangles. The following is the main result of this paper.

Theorem 3.1. *If G is T-equivalent to the twisted wheel W_{k_1,k_2} where $k_1 \ge k_2 \ge 3$, then G is isomorphic to W_{k_1,k_2}. That is, W_{k_1,k_2} is T-unique.*

The edges $v_{1,0}v_{2,k_2}$ and $v_{2,0}v_{1,k_1}$ are called *brim spoke-edges*, the edges $v_{1,0}v_{1,1}$ and $v_{2,0}v_{2,1}$ are called *brim non-spoke-edges* and the edge $e_0 = v_{1,0}v_{2,0}$ is called the *strap edge*.

We assume that $k_1+k_2+2 = n$ and $k_1 \ge k_2 \ge 3$. Then in W_{k_1,k_2}, there are n vertices, $2n-2$ edges, $n-2$ triangles, $n-2$ C_4's and $n-3$ C_4^+'s. Moreover, W_{k_1,k_2} contains no $K_{3,2}^+$ as a subgraph and is a graph in $\wp^{n-2}$. In W_{k_1,k_2}, each vertex and each edge except the strap edge is contained in a triangle.

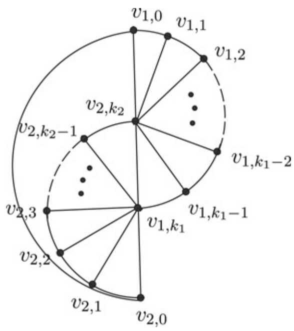


Fig. 3. The twisted wheel W_{k_1,k_2}.

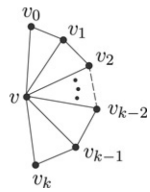


Fig. 4. A k-fan F_k.

Definition 3.2. A k-fan F_k is a maximal triangular k-path graph such that these k triangles share a common vertex v (see Fig. 4). The vertex v is called the *central vertex* of the k-fan. The edges incident with the center are called *spoke-edges* and the other edges are called *non-spoke-edges*. The edges vv_0 and vv_k are called *brim spoke-edges* of a k-fan, and the edges v_0v_1 and $v_{k-1}v_k$ are called *brim non-spoke-edges* of a k-fan.

de Mier and Noy [9] introduced a new polynomial, *rank-size generating polynomial*, as follows

$$F(G; x, y) = \sum_{S\subseteq E} x^{r(S)} y^{|S|}.$$

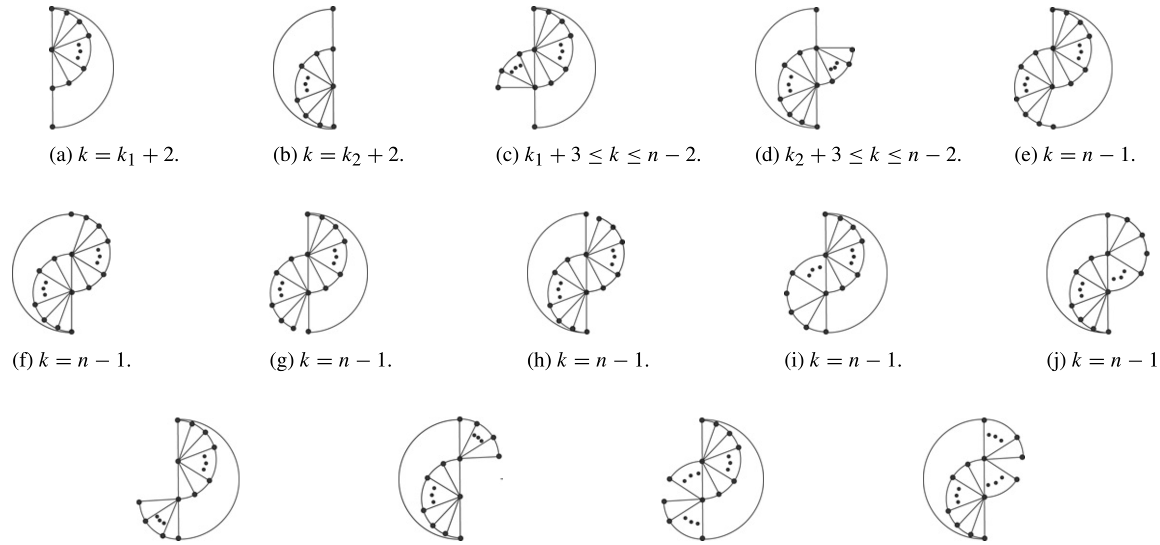
It is not hard to see that the coefficient of each monomial $x^{r(S)}y^{|S|}$ in $F(G; x, y)$ can be derived from $T(G; x, y)$, and vice versa. That is, these two polynomials contain exactly the same information about G. However, it is relatively easier to extract information from $F(G; x, y)$ than from $T(G; x, y)$. So the rank-size generating polynomial becomes a useful tool to mine invariant properties for $T(G; x, y)$ and thus to prove T-uniqueness property of many families of graphs. Following the same notion in [9], we use $[x^i y^j]F(G; x, y)$ to denote the coefficient of the monomial $x^i y^j$ in the polynomial $F(G; x, y)$.

Proposition 3.3. *For a twisted wheel W_{k_1,k_2} and $A \subseteq E(W_{k_1,k_2})$, the following holds*

(i) $[x^k y^{2k+i}]F(W_{k_1,k_2}; x, y) = 0$ *for $k \le n-2$ and $i \ge 0$;*
(ii) *If the strap edge $e_0 \notin A$, then the subgraphs induced by A contributing to the coefficient of $x^k y^{2k-1}$ are those subgraphs isomorphic to graphs in $\wp^{k-1}_{\max}$;*
(iii) *If the strap edge $e_0 \in A$, then the subgraphs induced by A contributing to the coefficient of $x^k y^{2k-1}$ are the three types of graphs shown in Fig. 5.*

Proof. Consider a subset $A \subseteq E(W_{k_1,k_2})$, $A = A_0 \cup A_0' \cup A_0''$, where $A_0 = \{e_0\}$ or $\varnothing$, A_0' is the set of edges in $E(W_{k_1,k_2}) - A_0$ contained in a cycle of $G[A - A_0]$, and A_0'' is the set of edges in $E(W_{k_1,k_2}) - A_0$ contained in no cycles of $G[A - A_0]$.

For all vertices in W_{k_1,k_2}, we give an ordering that is quite similar to the lexicographical ordering for their sub-indices: $(1, 0) < (1, 1) < (1, 2) < \cdots < (1, k_1 - 1) < (2, k_2) < (2, k_2 - 1) < \cdots < (2, 0) < (1, k_1)$. Let $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_m$ be the chordless cycles of $G[A_0']$ with the order of these cycles given by the minimum suffix of vertices of each cycle. Let $c_1, c_2, \ldots, c_m$ denote the sizes of these cycles respectively. Since $G[A_0']$ is a subgraph of W_{k_1,k_2}, every two chordless cycles share at most one edge.



(a) $k = k_1 + 2$. (b) $k = k_2 + 2$. (c) $k_1 + 3 \le k \le n - 2$. (d) $k_2 + 3 \le k \le n - 2$. (e) $k = n - 1$.

(f) $k = n - 1$. (g) $k = n - 1$. (h) $k = n - 1$. (i) $k = n - 1$. (j) $k = n - 1$.

(k) $k_1 + 3 \le k \le n - 2$. (l) $k_2 + 3 \le k \le n - 2$. (m) $k = n - 1$. (n) $k = n - 1$.

Fig. 5. Three types of subgraphs contributing to the coefficient of $x^k y^{2k-1}$.

Let f be the number of edges in A_0''. Define

$$\theta_i = \begin{cases} 1, & \text{if } i = 1; \\ 2, & \text{if } i \ge 2; \text{ moreover, } \mathscr{C}_i \text{ and } \mathscr{C}_{i-1} \text{ share an edge;} \\ 1, & \text{if } i \ge 2; \text{ moreover, } \mathscr{C}_i \text{ and } \mathscr{C}_{i-1} \text{ share no edges.} \end{cases}$$

$$\psi = \begin{cases} 0, & \text{if } e_0 \notin A; \\ 1, & \text{otherwise.} \end{cases}$$

$$\psi' = \begin{cases} 0, & \text{if } e_0 \notin A, \text{ or } e_0 \text{ is contained in a cycle of } G[A]; \\ 1, & \text{otherwise.} \end{cases}$$

Then we obtain the following two equations:

$$|A| = \sum_{i=1}^{m}(c_i - \theta_i + 1) + f + \psi = \sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + m + f + \psi. \tag{1}$$

$$r(A) = \sum_{i=1}^{m}(c_i - \theta_i) + f + \psi' = \sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f + \psi'. \tag{2}$$

Consider the edge subset contributing to the coefficient of $x^k y^{2k}$. Since $|A| = 2r(A)$, from Eqs. (1) and (2), we deduce that

$$\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f + 2\psi' = m + \psi. \tag{3}$$

If $\psi = 0$, then $\psi' = 0$, and $\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f = m$. Since $c_i \ge 3$, $\theta_i \le 2$ $(i \ge 2)$, $\theta_1 = 1$, and $f \ge 0$, we deduce that $\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f \ge m + 1$, which is impossible.

If $\psi = 1$, then clearly $\psi' = 0$ (otherwise, from Eq. (3), we have $\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f + 2 = m + 1$, which is a contradiction to $\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f \ge m + 1$). In addition, equality holds in $\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f \ge m + 1$ and therefore $c_i = 3$ $(1 \le i \le m)$, $\theta_i = 2$ $(2 \le i \le m)$, and $f = 0$, i.e. $A = E(W_{k_1,k_2})$. That is, in $F(W_{k_1,k_2}; x, y)$ the only edge subset contributing to the coefficient of $x^k y^{2k}$ is $E(W_{k_1,k_2})$, and the only monomial is $x^{n-1} y^{2n-2}$ with coefficient one. Moreover, $[x^k y^{2k+i}]F(W_{k_1,k_2}; x, y) = 0$ for $k \le n-2$ and $i > 0$, as $\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f \ge m+1$ implies $|A| \le 2r(A)$. Thus (i) holds.

Next consider the edge subset contributing to the coefficient of $x^k y^{2k-1}$ where $1 \le k \le n - 1$.

If $\psi = 0$, i.e., $e_0 \notin A$, then $\psi' = 0$. As $|A| = 2r(A) - 1$, by Eqs. (1) and (2), we deduce the following equation:

$$\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f - 1 = m. \tag{4}$$

As $\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f \geq m+1$, we deduce that $c_i = 3$ $(1 \leq i \leq m)$, $\theta_i = 2$ $(2 \leq i \leq m)$, $f = 0$, and $m+1 = k$. If $e_0 \notin A$, then $G[A]$ is a subgraph of W_{k_1,k_2} which is isomorphic to a graph in $\wp_{\max}^{k-1}$. Thus (ii) follows.

If $\psi = 1$, i.e., $e_0 \in A$, then

$$\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f - 1 + 2\psi' = m + 1. \tag{5}$$

As $\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f \geq m + 1$, we conclude that $\psi' = 0$, i.e., $e_0 \in A$, and e_0 is contained in a cycle of $G[A]$. From Eq. (5), one of the following three cases must hold.

Case 1. $c_i = 3\,(1 \leq i \leq m)$, $\theta_i = 2\,(2 \leq i \leq m)$ and $f = 1$.

In this case, in order to maintain the rank after adding e_0, either both edges $v_{1,0}v_{2,k_2}$ and $v_{2,0}v_{1,k_1}$ are in A, or one of the brim non-spoke-edges is in A. Thus $G[A]$ is a subgraph of W_{k_1,k_2} isomorphic to one of the graphs shown in Fig. 5(a.1)–(a.8).

Case 2. There exists t $(1 \leq t \leq m)$ such that $c_t = 4$; moreover, $f = 0$, $c_i = 3$, and $\theta_i = 2$ for all $i \neq t$ $(2 \leq i \leq m)$.

In this case, $k = n - 1$ and $G[A]$ is the subgraph obtained by deleting a spoke-edge (except the two brim spoke-edges) of W_{k_1,k_2}, as shown in Fig. 5(b.1) and (b.2).

Case 3. For some t $(2 \leq t \leq m)$ we have that $\theta_t = 1$; moreover, for all $i \neq t$, we have that $\theta_i = 2$, $c_i = 3$ $(1 \leq i \leq m)$, and $f = 0$.

In this case, $G[A]$ is a subgraph of W_{k_1,k_2} isomorphic to one of the graphs shown in Fig. 5(c.1)–(c.4). This completes the proof of the proposition. □

Remark. In W_{k_1,k_2} $(3 \leq k_2 \leq k_1)$, if $k \leq k_2 + 1$, then the subgraphs induced by A contributing to the coefficient of $x^k y^{2k-1}$ are those subgraphs of W_{k_1,k_2} isomorphic to graphs in $\wp_{\max}^{k-1}$ and the total number of such subgraphs is $n - k$. If $k \geq k_2 + 2$, then the subgraphs induced by A contributing to the coefficient of $x^k y^{2k-1}$ are those subgraphs isomorphic to graphs in $\wp_{\max}^{k-1}$ and the subgraphs in Fig. 5, and there are at least $n - k + 1$ such graphs in total.

Assume that a graph G is T-equivalent to W_{k_1,k_2}. Our proof to show that G is isomorphic to W_{k_1,k_2} includes the following steps: (1) there are no cycles of length p $(4 \leq p \leq n-2)$ in $TR(G)$ (Lemma 3.5); (2) there is no triangle in $TR(G)$ (Lemma 3.6); (3) $TR(G)$ is a special tree, i.e., a path; (4) $TR(G)$ is a path of length $n - 2$. Once these claims are confirmed, the main result follows easily.

Lemma 3.4. *If a graph G is T-equivalent to W_{k_1,k_2}, then G contains no $C^p \in \mathcal{C}^p$ as a subgraph for all p, $4 \leq p \leq n - 2$.*

Proof. Suppose that G contains C^p as a subgraph for some p, $4 \leq p \leq n - 2$. Then by Proposition 2.4(ii), $[x^{p-i}y^{2p}]F(G; x, y) \geq 1$ for some $i \geq 0$. As G is T-equivalent to W_{k_1,k_2}, by Proposition 3.3(i), we conclude that for all $i \geq 0$ and $4 \leq p \leq n - 2$,

$$[x^{p-i}y^{2(p-i)+2i}]F(G; x, y) = [x^{p-i}y^{2(p-i)+2i}]F(W_{k_1,k_2}; x, y) = 0.$$

This is a contradiction. □

Lemma 3.5. *If a graph G is T-equivalent to W_{k_1,k_2} and G contains no $K_{q,2}^+$ as a subgraph $(q \geq 4)$, then $TR(G)$ does not contain any cycles, i.e., $TR(G)$ is acyclic.*

Proof. We proceed by contradiction. Let $C_p = v_1v_2v_3 \cdots v_pv_1$ be a shortest cycle of $TR(G)$ such that $4 \leq p \leq n-2$. By Lemma 3.4, G contains no subgraph isomorphic to any graph in $\mathcal{C}^p$. We conclude that C_p has at least one chord, say, v_1v_s. Let $S_1 = v_1v_2 \cdots v_sv_1$ and $S_2 = v_1v_p \cdots v_sv_1$. Clearly, both S_1 and S_2 have at least three edges. Since C_p is a shortest cycle in $TR(G)$ such that $4 \leq p \leq n - 2$, we conclude that $|S_1| = |S_2| = 3$. Thus $TR(G)$ contains C_4^+ as a subgraph. Now it is straightforward to show that G contains either $K_{4,2}^+$ or K_4 as a subgraph. By our assumption that G contains no $K_{4,2}^+$ as a subgraph, we conclude that G contains K_4 as a subgraph. However, by Theorem 1.1(v), G contains no K_4 in W_{k_1,k_2}. The last contradiction completes the proof of the lemma. □

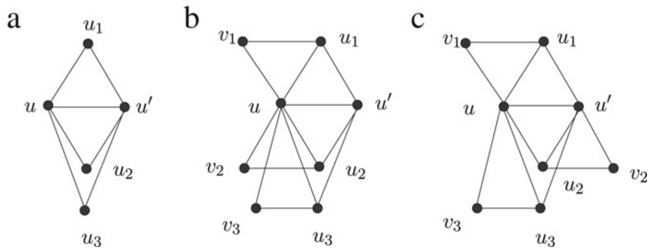


Fig. 6. G_0, G_1 and G_2.

Lemma 3.6. *If a graph G is T-equivalent to the twisted wheel W_{k_1,k_2} where $k_1 \geq k_2 \geq 3$, then no edge in G is contained in three or more triangles.*

Proof. Theorems 1.1 and 1.2 as well as the remark after Theorem 1.2 yield the following facts:

(1) G is simple and 2-connected;
(2) $|V(G)| = n$, $|E(G)| = 2n - 2$;
(3) The numbers of K_3's and K_4's are $n - 2$ and 0, respectively;
(4) The numbers of C_4's and C_4^+'s are $n - 2$ and $n - 3$, respectively;
(5) G is 3-edge-connected and the number of 3-element bonds is $n - 2$.

Let τ_i be the number of edges of G contained in exactly i triangles, $i \geq 0$. In the following, we count the number of occurrences where edges are contained in triangles of G: $\sum_{i=3}^{n-2} i\tau_i$ is the number of occurrences where edges are contained in at least three triangles; $2\left(n - 3 - \sum_{i=3}^{n-2} \tau_i \binom{i}{2}\right)$ is the number of occurrences where edges are contained in exactly two triangles and $\left[2n - 2 - \sum_{i=3}^{n-2} \tau_i - \left(n - 3 - \sum_{i=3}^{n-2} \tau_i \binom{i}{2}\right)\right]$ is the number of occurrences where edges are contained in at most one triangle. Then

$$3(n-2) \leq \sum_{i=3}^{n-2} i\tau_i + 2\left(n - 3 - \sum_{i=3}^{n-2} \tau_i \binom{i}{2}\right) + \left[2n - 2 - \sum_{i=3}^{n-2} \tau_i - \left(n - 3 - \sum_{i=3}^{n-2} \tau_i \binom{i}{2}\right)\right]. \tag{6}$$

So we have $\sum_{i=3}^{n-2} \left[\binom{i}{2} - (i-1)\right] \tau_i \leq 1$. Therefore $\tau_3 \leq 1$, and $\tau_i = 0$ for $i \geq 4$. Thus, G contains no $K_{q,2}^+$ $(q \geq 4)$ as a subgraph, and the number of $K_{3,2}^+$'s in G is $\tau_3 \leq 1$.

Next we count the number of subgraphs of G contributing to the coefficient of x^4y^7. There are four possible subgraphs: (a) $K_{3,2}^+$; (b) a 3-fan; (c) a complete subgraph K_4 plus an extra edge; (d) $K_{3,2}$ with an extra edge joining two vertices in the partite set of order three. By Lemma 3.4, G contains no C^p for all p in $\{4, \ldots, n-2\}$. Moreover, G contains no K_4, and all but one 4-element cycle contain a chord. Hence the subgraphs (c) and (d) do not occur. We conclude that the possible subgraphs contributing to the coefficient of x^4y^7 can only be $K_{3,2}^+$'s and 3-fans. Let c be the number of subgraphs of G isomorphic to 3-fan. Then $\tau_3 + c = [x^4y^7]F(G; x, y) = n - 4$. Next we show that $\tau_3 = 0$.

Suppose not, that is, $\tau_3 = 1$. Then $c = n - 5$ and G contains exactly one $K_{3,2}^+$ as a subgraph denoted by G_0 (see Fig. 6(a)). By Lemma 3.5, $TR(G)$ contains exactly one cycle and its length is three. From Eq. (6), every edge of G is contained in a triangle and thus every vertex of G is in a triangle too. In G_0, let $e_0 = uu'$, $e_i = uu_i$ and $e_i' = u'u_i$ $(i = 1, 2, 3)$, and denote the triangles uu_iu' by t_i. Let r be the number of triangles of $G \setminus e_0$ containing edges e_i or e_i' for $i = 1, 2, 3$. Clearly $0 \leq r \leq 6$. Note that a C_4^+ in G corresponds to an edge of $TR(G)$, and a 3-fan of G corresponds to a 2-path of $TR(G)$.

Suppose that $r = 6$. Let $H_1 = G \setminus e_0$. Then there are $n - 5$ triangles, $n - 12C_4^+$'s and at most $n - 20$ 3-fans in H_1 and $TR(H_1)$ is acyclic. Thus $TR(H_1)$ is a forest with $n - 5$ vertices and 7 components, but by Lemma 2.6, there are at least $n - 19$ 2-paths in $TR(H_1)$, a contradiction. We conclude that $0 \leq r \leq 5$.

Case 1. There exists a triangle t_i $(1 \leq i \leq 3)$, say t_3, in which the edges e_i and e_i' are not contained in any other triangle.

Let $H_2 = G \setminus e_3$. Then there are $n - 3$ triangles, $n - 5$ C_4^+'s and $n - 5 - r$ 3-fans in H_2 and $TR(H_2)$ is acyclic, i.e., $TR(H_2)$ is a forest with $n - 3$ vertices and 2 components. Thus H_2 is a graph of $\mathcal{F}_2^{n-3}$. Let $H_2 = T^{n_1} \cup T^{n_2}$, where

$T^{n_1} \in \Gamma^{n_1}, T^{n_2} \in \Gamma^{n_2}$ and $n_1 + n_2 = n - 3$. By Proposition 2.4(i), $|E(\hat{H}_2)| = (2n_1 + 1) + (2n_2 + 1) = 2n - 4$. Since e_3 and e'_3 are contained in t_3 only, we deduce that $e_3, e'_3 \notin E(\hat{H}_2)$. Thus $E(G) = E(\hat{H}_2) \cup \{e_3, e'_3\}$ and $V(G) = V(\hat{H}_2) \cup \{u_3\}$. As G is 3-edge-connected, $\delta(G) \geq 3$. Thus in H_2, there is at least one edge e''_3 ($e''_3 \notin \{e_3, e'_3\}$) incident with u_3. Since each edge of G is contained in a triangle, $e''_3 \in E(\hat{H}_2)$ and thus $u_3 \in V(\hat{H}_2)$. Therefore $V(G) = V(\hat{H}_2)$. As G is 2-connected and $\delta(G) \geq 3$, we conclude that $|V(G)| = |V(\hat{H}_2)| \leq (n_1 + 2) + (n_2 + 2) - 2 = n - 1$, a contradiction.

Case 2. For every triangle t_i ($i = 1, 2, 3$), at least one of the edges e_i and e'_i is contained in another triangle.

In this case, $r \geq 3$. Since $r \leq 5$, there exists an edge, say e'_3, contained in no other triangles. Let $H_3 = G \setminus e'_3$. Then there are $n - 3$ triangles and $n - 6$ C_4^+'s in H_3. Moreover, the number of 3-fans decreased by at least $r + 1$ after deleting the edge e'_3 from G. Therefore there are at most $n - 6 - r$ 3-fans. Clearly $TR(H_3)$ is acyclic. Thus $TR(H_3)$ is a forest with $n - 3$ vertices, three components and at most $n - 6 - r$ 2-paths. By Lemma 2.6, there are at least $n - 9$ 2-paths in $TR(H_3)$. Thus, $n - 6 - r \geq n - 9$, or $r \leq 3$. Hence, in this case, $r = 3$ and G contains G_1 or G_2 shown in Fig. 6(b) and (c) as a subgraph and $TR(H_3)$ has exactly $n - 9$ 2-paths. In addition, the number of 3-fans decreased exactly by four after deleting e'_3 from G. Thus, both uv_3 and u_3v_3 are contained only in the triangle uu_3v_3. Therefore, H_3 is a graph of F_3^{n-3} and it can be written as $H_3 = P^{n_1} \cup P^{n_2} \cup P^{n_3}$, $P^{n_i} \in \wp^{n_i}$ ($i = 1, 2, 3$) and one of P^{n_i}'s contains exactly one triangle, say P^{n_2}. As each edge and each vertex of G is contained in a triangle, it is easy to see that $|E(H_3)| = |E(\hat{H}_3)|$ and $|V(H_3)| = |V(\hat{H}_3)|$. Since $|E(\hat{H}_3)| = 2n - 3$, we have that $E(G) = E(\hat{H}_3) \cup \{e'_3\}$ and $V(G) = V(\hat{H}_3) = V(P^{n_1}) \cup V(P^{n_2}) \cup V(P^{n_3})$.

From the structures of G_1 and G_2, we may assume that $u \in V(P^{n_1}) \cup V(P^{n_2})$. Since G is 2-connected and $\delta(G) \geq 3$, we deduce that $|V(G)| = |V(\hat{H}_3)| \leq |V(P^{n_1})| + |V(P^{n_2})| - 1 + |V(P^{n_3})| - 3 \leq n - 1$, a contradiction again. This completes the proof of Case 2 and thus we show that $\tau_i = 0$ for $i \geq 3$, i.e., no edge is contained in three or more triangles. $\square$

Combining Lemmas 3.4–3.6, we obtain the following.

Corollary 3.7. *If a graph G is T-equivalent to W_{k_1,k_2}, then G contains no $C^p \in \mathcal{C}^p$ as a subgraph for $3 \leq p \leq n-2$. Furthermore, $TR(G)$ is acyclic.*

Lemma 3.8. *Suppose that G is a graph satisfying the following conditions*

(i) *G is 3-edge-connected,*
(ii) *$\hat{G} \in \wp_{\max}^{n-2}$ where $n \geq 8$, and*
(iii) *$E(G) = E(\hat{G}) \cup \{e\}$, where e is an edge joining the two vertices of degree 2 in $\hat{G}$.*

Then every 3-element bond of G is trivial.

Proof. Suppose not. Then there exists a non-trivial 3-element bond $B = \{f_1, f_2, f_3\}$ which induces a partition (A, A') of $V(G)$. Let $f_i = v_iv'_i$ ($1 \leq i \leq 3$), where $v_1, v_2, v_3 \in A$, and $v'_1, v'_2, v'_3 \in A'$. As B is non-trivial, both A and A' have at least two vertices.

Case 1. Each of f_1, f_2, f_3 is contained in a triangle, say $f_i \in t_i = v_iv'_iu_i$. Then $u_1 \in \{v_2, v'_2, v_3, v'_3\}$; otherwise, either v_1u_1 or v'_1u_1 would connect A to A', a contradiction as B is a bond with exactly three edges. Without loss of generality, assume that $u_1 = v_2$. Then $v_1v_2 \in E(G)$ and $v'_1 = v'_2$. Similarly, $u_3 \in \{v_1, v_2, v'_1\}$. If $v_3 \notin \{v_1, v_2\}$, then $v'_3 = v'_1$ as B is a bond with exactly three edges. But then B is a trivial bond, a contradiction. Therefore, $v_3 \in \{v_1, v_2\}$. Without loss of generality, assume that $v_3 = v_2$. Then $u_3 = v'_1$, and $v'_1v'_3 \in E(G)$.

Denote Q_2 the subgraph induced by the set $\{v_1, v_2, v'_1, v'_3\}$. Then $Q_2 \cong C_4^+$. In Q_2, each of f_1, f_3 is contained in exactly one triangle, and f_2 is contained in exactly two triangles. Since $G \in \wp^{n-2}$ and $n \geq 8$, at least one of the edges $v_1v_2, v'_1v'_3$, say the former, is contained in another triangle. Let $w_3v_1v_2$ be the new triangle sharing an edge v_1v_2 with the triangle $v_1v_2v'_1$. Then it is easy to see that $w_3 \in A$. Thus $Q_3 = Q_2 \cup \{v_1w_3, v_2w_3\}$ is a 3-fan. Similarly, if Q_i is already constructed ($i \leq n - 3$), then Q_{i+1} is constructed by adding a new vertex w_{i+1} and two new edges $\{w_{i+1}w'_i, w_{i+1}w''_i\}$, where w'_i and w''_i are contained in the same set in the partition (A, A'), and the edge $w'_iw''_i$ is contained in exactly one triangle of Q_i. Clearly, w_{i+1} is in the same set as w'_i and w''_i in the partition. In this way, we see that $Q_{n-2} \in \wp_{\max}^{n-2}$. In Q_{n-2}, there are two vertices of degree two contained in A and A', respectively. Condition

(iii) implies that there is an edge joining these two vertices, which contradicts the assumption that f_1, f_2, f_3 is a non-trivial 3-element bond.

Case 2. One of the edges in $\{f_1, f_2, f_3\}$ is not contained in any triangle, say f_1.

Then f_1 is the edge e given in Condition (iii). By our assumption, each of f_2 and f_3 is contained in a triangle. As B is a bond with exactly three edges, it is easy to see that either $v_2v_3 \in E(G)$ and $v_2' = v_3'$, or $v_2 = v_3$ and $v_2'v_3' \in E(G)$. Without loss of generality, assume that the former occurs. Now v_2v_3 must be contained in another triangle. Using a similar method as in Case 1 to construct G, finally there must be a vertex $w_{n-2} \in A$ of degree 2. Since f_1 is contained in no triangle, it must be the edge joining the two vertices of degree 2. Clearly these two vertices must be v_2' and w_{n-2}, respectively. Thus $v_1 = w_{n-2}$ and $v_1' = v_2'$, which is a contradiction to the assumption that $\{f_1, f_2, f_3\}$ is a non-trivial 3-element bond. □

With the preparation above, we are ready to prove Theorem 3.1, our main result.

Proof of Theorem 3.1. From Eq. (6) in the proof of Lemma 3.6, among $2n-2$ edges of G, $n-3$ edges are contained in exactly two triangles, n edges are contained in exactly one triangle and one edge is contained in no triangles. By Corollary 3.7, $TR(G)$ is acyclic. Moreover, there are $n-2$ triangles, $n-3C_4^+$'s in G. Hence $TR(G)$ is a tree of $n-2$ vertices. Furthermore, inheriting the notion from the proof of Lemma 3.6, we have that $\tau_3 = 0$ and $c = n-4$, that is, there are exactly $n-4$ 3-fans. Thus $TR(G)$ is a path of length $n-2$, or $G \in \wp^{n-2}$.

Since G is 3-edge-connected, $|V(G)| = n$, $|E(G)| = 2n-2$ and every vertex is contained in a triangle (i.e., $V(\hat{G}) = V(G)$), we deduce that G satisfies the three conditions of Lemma 3.8. Therefore every 3-element bond of G is trivial. We conclude that there are $n-2$ vertices of degree three in G, and exactly two vertices of degree greater than three. Because $\hat{G} \in \wp^{n-2}_{\max}$, there must exist an i-fan for some $i \le n-3$ (otherwise, if $i = n-2$, joining the two vertices of degree two in G will increase the number of triangles of G by one). Let F_{θ_1} be a θ_1-fan with the maximum number of triangles of G, where $3 \le \theta_1 \le n-3$. Then there must be another triangle t sharing a brim non-spoke-edge of F_{θ_1}. Now $V(F_{\theta_1}) \cup V(t)$ has two vertices, denoted by u and v, of degree greater than three. Since there are exactly two vertices of degree greater than three in G, we conclude that G contains exactly two maximal fans, namely the θ_1-fan and another θ_2-fan. Moreover, u and v are the central vertices of the two fans. Clearly, $\theta_2 \ge 4$ (otherwise, there are two C_4's containing no chord). Now it is straightforward to see that G is isomorphic to $W_{k_1',k_2'}$, where $k_1' = \theta_1 - 1$, $k_2' = \theta_2 - 1$ and $\theta_1', \theta_2' \ge 4$. By the above proof, $k_1' \ge k_2' \ge 3$.

Next we complete the proof of the theorem by showing that $k_1' = k_1$ and $k_2' = k_2$. Suppose that $k_2 \ne k_2'$, say, $k_2 < k_2'$. Then $k_2 < k_2' \le k_1' < k_1$. Consider the coefficient of $x^{k_2+2}y^{2k_2+3}$. By the remark after Proposition 3.3, $[x^{k_2+2}y^{2k_2+3}]F(W_{k_1,k_2}; x, y) \ge n-k_2-1$. However, since $k_2+2 \le k_2'+1$, $[x^{k_2+2}y^{2k_2+3}]F(W_{k_1',k_2'}; x, y) = n-k_2-2$, a contradiction. Therefore, $k_2' = k_2$ and thus $k_1' = k_1$. This completes the proof of the theorem. □

It is well-known that the chromatic polynomial and the flow polynomial are two important evaluations of the Tutte polynomial (see, for example, [2]). To conclude the paper, we would like to mention that it would be interesting to know if twisted wheels can be determined by their chromatic polynomials or flow polynomials alone or both polynomials together.

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References

[1] J. Benashski, R. Martin, J. Moore, L. Traldi, On the β-invariant for graphs, Congr. Numer. 109 (1995) 211–221.

[2] B. Bollobás, Modern Graph Theory, Springer-Verlag, Berlin Heidelberg, 1998.

[3] B. Bollobás, L. Pebody, O. Riordan, Contraction-deletion invariant for graphs, J. Combin. Theory Ser. B 80 (2000) 320–345.

[4] J.E. Bonin, A. de Mier, T-uniqueness of some families of k-chordal matroids, in: Tutte polynomial, Adv. Appl. Math. 32 (1–2) (2004) 10–30 (special issue).
[5] T. Brylawski, J.G. Oxley, The Tutte polynomial and its applications, in: N. White (Ed.), Matroid Applications, Cambridge University Press, Cambridge, 1992.
[6] Y. Duan, H. Wu, Q. Yu, On chromatic and flow polynomial unique graphs, Discrete Appl. Math. (2007), doi:10.1016/j.dam.2007.10.010.
[7] J.S. Kuhl, The Tutte Polynomial and the generalized Peterson graph, Australas. J. Combin. (in press).
[8] A. Márquez, A. de Mier, M. Noy, M.P. Revuelta, Locally grid graphs: Classification and Tutte uniqueness, Discrete Math. 266 (2003) 327–352.
[9] A. de Mier, M. Noy, On graphs determined by their Tutte polynomials, Graphs Combin. 20 (2004) 105–119.
[10] A. de Mier, M. Noy, Tutte uniqueness of line graphs, Discrete Math. 301 (2005) 57–65.
[11] J.G. Oxley, H. Wu, Matroids and graphs with few non-essential elements, Graphs Combin. 16 (2) (2000) 199–229.
[12] W.T. Tutte, A contribution to the theory of chromatic polynomials, Canad. J. Math. 6 (1954) 80–91.
[13] H. Wu, Connectivity for matroids and graphs, Ph.D. Dissertation, Louisiana State University, 1994.