On Superconnectivity of $(4, g)$-Cages with Even Girth

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Abstract

A (k, g)-*cage* is a k-regular graph with girth g that has the fewest number of vertices. It has been conjectured [Fu, Huang, and Rodger, Connectivity of cages, J. Graph Theory 24 (1997), 187-191] that all (k, g)-cages are k-connected for $k \geq 3$. A connected graph G is said to be *superconnected* if every minimum cut-set S is the neighborhood of a vertex of minimum degree. Moreover, if $G - S$ has precisely two components, then G is called *tightly superconnected*. It was shown [Xu, Wang, and Wang, On the connectivity of $(4, g)$-cages, Ars Combin 64 (2002), 181-192] that every $(4, g)$-cage is 4-connected. In this paper, we prove that every $(4, g)$-cage is tightly superconnected when g is even and $g \geq 12$.

Key words: cage, girth, superconnected, tightly superconnected

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1 Introduction

Throughout this paper, only undirected simple graphs are considered. Unless otherwise defined, we follow [2] for terminologies and definitions.

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $d_G(u, v)$ denote the distance between two vertices $u, v \in V(G)$. For vertex sets $T_1, T_2 \subseteq V(G)$, $E(T_1, T_2)$ is the set of the edges between T_1 and T_2, and $d(T_1, T_2) = d_G(T_1, T_2) = \min\{d(t_1, t_2) \mid t_1 \in T_1, t_2 \in T_2\}$ denotes the *distance* between T_1 and T_2. Let $|P_k(T_1, T_2)|$ and $|P_{\leq k}(T_1, T_2)|$ denote the number of paths of length k and no more than k from T_1 to T_2, respectively. For $S \subset V(G)$, $G[S]$ denotes the subgraph induced by vertex set S, and $G - S$ is the subgraph of G obtained by deleting the vertices in S and all the edges incident with them. The set of vertices which are at distance r from S in G is denoted by $N_r(S) = \{v \in V(G) \mid d(v, S) = r\}$, where r is an integer. We use $N(S)$ instead of $N_1(S)$. The length of a shortest cycle in G is called the *girth* of G, denoted by $g(G)$. The *diameter* of G is the maximum distance between any two vertices in G. By *connecting two vertices*, we mean joining the two vertices by an edge and *connecting a vertex* x to a vertex set R means joining x to every vertex in R.

A k-regular graph with girth g is called a (k, g)-*graph*. A (k, g)-*cage* is a (k, g)-graph with the fewest number of vertices for given k and g. We use $f(k, g)$ to denote the number of vertices in (k, g)-cages. A cut-set X of G is called a *trivial cut-set* if X is the neighborhood of a vertex of minimum degree. A k-connected (or k-vertex-connected) graph G is called *superconnected* if for every vertex cut-set $S \subseteq V(G)$ with $|S| = k$, S is a trivial cut-set. Moreover, if $G - S$ has precisely two components, then G is called *tightly superconnected*. Provided that a non-trivial cut-set exists, the *superconnectivity* of G is denoted by $\kappa_1 = \kappa_1(G) = \min\{|X| \mid X \text{ is a non-trivial cut-set}\}$. The edge-superconnectivity λ_1 is defined similarly.

Cages were introduced by Tutte in 1947, and have been extensively studied; we refer the reader to the survey [17] for more detailed information. Recently, due to the importance of cage connectivity in the design of efficient and reliable networks, several researchers have studied the connectivity of cages; for example [3, 4, 6, 10, 9, 11, 12]. Fu, Huang and Rodger [6] conjectured that (k, g)-cages are k-connected. Daven and Rodger [3], and independently

Jiang and Mubayi [7], proved that all (k,g)-cages are 3-connected for $k \geq 3$. Additionally, it was proved that every $(4,g)$-cage is 4-connected [18]. For $k \geq 4$, Marcote et al. [14] showed that (k,g)-cages with $g \geq 10$ are 4-connected. Recently, Lin, Miller and Balbuena [9] proved that all (k,g)-cages are r-connected with $r \geq \sqrt{k+1}$ for $g \geq 7$ odd; Lin et al. [8] also proved that all (k,g)-cages with even girth are $(r+1)$-connected, where r is the largest integer such that $r^3 + 2r^2 \leq k$. In this paper, we show that every $(4,g)$-cage with even girth $g \geq 12$ is tightly superconnected.

For the edge-connectivity of (k,g)-cages, Wang, Xu and Wang [16] showed that (k,g)-cages are k-edge-connected when g is odd, and subsequently, Lin, Miller and Rodger [11] proved that (k,g)-cages are k-edge-connected when g is even. Recently, Lin et al. [10, 13] proved that (k,g)-cages are edge-superconnected.

2 Main Result

In this section, we prove that every $(4,g)$-cage with even girth $g \geq 12$ is tightly superconnected. To show our main result, we use the following known results.

Theorem 1. ([5, 6]) *Let $k \geq 2$ and $g \geq 3$ be integers. The following statements hold:*

(1) *$f(k,g) < f(k,g+1)$;*

(2) *if D is the diameter of a (k,g)-cage, then $D \leq g$.*

Lemma 1. ([4]) *Let G be a (k,g)-cage with $k \geq 3$ and $g \geq 7$. If $S \subseteq V(G)$ and the diameter of $G[S]$ is at most $\lfloor g/2 \rfloor - 2$, then $G - S$ is 2-connected.*

For edge-connectivity, Tang et al. [15] conjectured the following:

Conjecture 1. ([15]) *Every (k,g)-cage of odd girth $g \geq 5$ has $\lambda_1 = 2k-2$.*

Lu et al. [12] showed the following result which supports Conjecture 1.

Lemma 2. ([12]) *Every $(4,g)$-cage of girth $g \geq 5$ has $\lambda_1 = 6$.*

The following lemma is also required for the proof of our main result.

Lemma 3. ([1]) *Let G be a $(4, g)$-cage with even girth $g \geq 6$. Assume that $X \subseteq V$ is a minimum non-trivial cut-set such that $|X| \leq 4$, and let C be a connected component of $G - X$. Then there exists a vertex $u \in V(C)$ such that $d(u, X) \geq g/2 - 1$.*

To prove that every $(4, g)$-cage G of even girth $g \geq 12$ is tightly superconnected, we use contrapositive arguments. We assume that there exists a non-trivial cut-set $|S| = 4$ in G; since $\lambda_1 = 6$, this implies that $G - S$ contains only two components C_1 and C_2. Let C_1 be the smaller one. Using C_1, we construct a $(4, g')$-graph of order less than $|V(G)|$, with $g' \geq g$, which contradicts Theorem 1.

By Theorem 1 and Lemma 3, we have $g/2 - 1 \leq \max\{d(v, S) \mid v \in V(C_1)\} \leq g/2 + 1$. Denote $S = \{s_1, s_2, s_3, s_4\}$. In order to present our main results, we first present two observations. In the following, we assume that all vertices in $G[S]$ are independent, i.e., $d_{G[S]}(s_i) = 0$, $i = 1, 2, 3, 4$.

Observation 1. *S can be partitioned into two disjoint vertex subsets S_1 and S_2 such that $d_G(S_1, S_2) \geq 3$, where $|S_1| = |S_2| = 2$.*

Proof. Let t be a vertex of $G - S$. Since S is a non-trivial cut-set, we have $|N(t) \cap S| \leq 3$. Furthermore, if $|N(t) \cap S| = 3$, then $G - ((N(t) \cap S) \cup t)$ contains a cut-vertex, which is a contradiction to Lemma 1. So $|N(t) \cap S| \leq 2$.

Let s be a vertex in S. We claim that there is at most one vertex in $S - s$ at distance two from s in G. Otherwise, suppose there are two distinct vertices t_1 and t_2 in $G - S$ and two vertices s' and s'' in $S - s$ such that st_1s' and st_2s'' are two paths of length two. Then $G[\{s, s', s'', t_1, t_2\}]$ is a graph of diameter four, and so $G - \{s, s', s'', t_1, t_2\}$ is 2-connected by Lemma 1, since $g \geq 12$. But, in fact, $G - \{s, s', s'', t_1, t_2\}$ contains a cut-vertex, a contradiction. Hence S can be partitioned into two disjoint vertex subsets, say $\{s_1, s_2\}$ and $\{s_3, s_4\}$, such that $d_G(\{s_1, s_2\}, \{s_3, s_4\}) \geq 3$. □

Observation 2. *Let $u \in V(C_1)$, $N(u) = \{u_1, u_2, u_3, u_4\}$ and $W_i = N(u_i) - u = \{u_{i1}, u_{i2}, u_{i3}\}$ $(i = 1, 2, 3, 4)$. Suppose that for each $s_i \in S$ there exists $T_j \subseteq W_j$ such that $|N(s_i) \cap V(C_2)| =$*

$|T_j|$ *and* $d(T_j, S) \geq g/2-1$. *If* T_1, T_2, T_3 *and* T_4 *are mutually disjoint, then* G *is not a* $(4, g)$-*cage.*

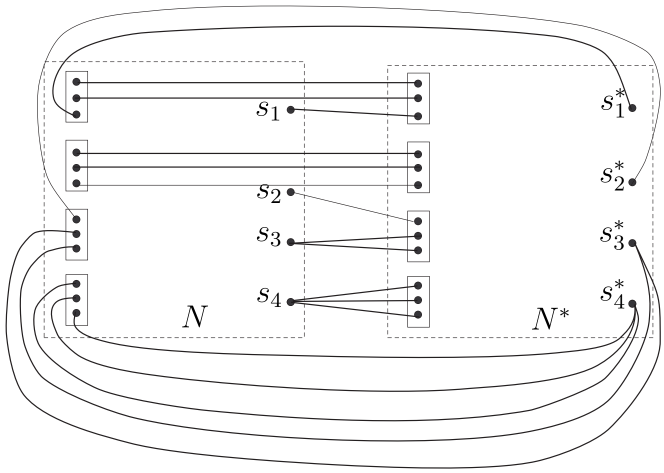

Figure 1: Illustration of the construction in Observation 2.

Proof. Let $N = G[(C_1 \cup S) - (N(u) \cup u)]$ and let N^* be a copy of N. For every $v \in V(N)$, let v^* denote the corresponding vertex in the copy of N^*. Now we construct a 4-regular graph H (see Figure 1) of girth at least g by using N and N^* and adding the following edges:

(a) connect s_i to T_i^* and s_i^* to T_i for $i = 1, 2, 3, 4$;

(b) connect r to r^*, where r is of degree three after the operation (a).

It is clear that any cycle entirely contained in either N or N^* has length at least g. Any new cycle $\mathcal{C}$ created in H must contain at least two new edges in (a) and (b). If $\mathcal{C}$ contains two edges in (a), then $\mathcal{C}$ has length at least g, since $d(T_i, S) \geq g/2-1$ for all i. If $\mathcal{C}$ contains two edges in (b), then its length is at least $2(g-4)+2 > g$, since $g \geq 12$. Finally, if $\mathcal{C}$ contains one edge in (a) and one edge in (b), then its length is at least $(g/2-1)+(g-4)+2 > g$. Thus H is a $(4, g')$-graph with smaller order than G, where $g' \geq g$, a contradiction to Theorem 1. $\square$

We now prove several lemmas, based on $\max\{d(v, S) \mid v \in V(C_1)\}$ and the degree distributions of the vertices of S in the component C_2.

Lemma 4. *If* $\max\{d(v,S) \mid v \in V(C_1)\} = g/2-1$, $|N(s_1)\cap V(C_2)| = |N(s_2)\cap V(C_2)| = 1$ *and* $|N(s_3)\cap V(C_2)| = |N(s_4)\cap V(C_2)| = 2$, *then G is not a $(4,g)$-cage.*

Proof. Let $d(u,S) = g/2-1$, $N(u) = \{u_1,u_2,u_3,u_4\}$ and $W_i = N(u_i) - u = \{u_{i1},u_{i2},u_{i3}\}$ $(i = 1,2,3,4)$. Then $g/2-3 \le d(W_i,S) \le g/2-2$. Due to the girth requirement, any two paths of length $g/2-3$ from $N_2(u)$ to S can not share the same end vertex in S. Hence $|P_{g/2-3}(N_2(u),S)| \le 4$. So we only need to consider the following three cases.

Case 1. $3 \le |P_{g/2-3}(N_2(u),S)| \le 4$.

Since no path of length at most $g/2-2$, from $N_2(u)$ to S, can share the same vertex in $N(s_i)\cap V(C_1)$, we have $|P_{g/2-2}(N_2(u),s_i)| \le 3$, as $2 \le |N(s_i)\cap V(C_1)| \le 3$. If $|P_{g/2-3}(N_2(u),S)| = 3$, then $|P_{g/2-2}(N_2(u),S)| \le 3$. If $|P_{g/2-3}(N_2(u),S)| = 4$, then there are no paths of length $g/2-2$ from $N_2(u)$ to S. As $|N_2(u)| = 12$, there are at least six vertices in $N_2(u)$ at distance $g/2-1$ from S. Since $g/2-2 \le d(W_i,S) \le g/2-3$, there exist two distinct vertices $r_1, r_2 \in N_2(u)$ such that $d(r_i,S) \ge g/2-1$ and two vertex sets, say $T_1 \subseteq W_1$ and $T_2 \subseteq W_2$, such that $d(T_i,S) \ge g/2-1$, where $r_1 \notin T_i$, $r_2 \notin T_i$ and $|T_i| = 2$ for $i = 1,2$. Hence G is not a $(4,g)$-cage by Observation 2.

Case 2. $P_{g/2-3}(N_2(u),S) = 2$ *and* $P_{g/2-3}(W_i,S) \le 1$ $(i = 1,2,3,4)$.

Subcase 2.1. The two paths of length $g/2-3$ are from $N_2(u)$ to $\{s_1,s_2\}$.

Without loss of generality, assume that $d(u_{11},s_1) = d(u_{21},s_2) = g/2-3$. Then $d(N_2(u)-\{u_{11},u_{21}\},\{s_1,s_2\}) \ge g/2-1$. $|N(s_3)\cap V(C_1)| + |N(s_4)\cap V(C_1)| = 4$, so $|P_{g/2-2}(N_2(u)-\{u_{11},u_{21}\},S)| \le 4$. Hence there are at least six vertices in $N_2(u)$ at distance $g/2-1$ from S. Using a similar argument as in Case 1, it follows that G is not a $(4,g)$-cage.

Subcase 2.2. The two paths of length $g/2-3$ are from $N_2(u)$ to $\{s_3,s_4\}$.

Suppose $d(u_{31},s_3) = d(u_{41},s_4) = g/2-3$, then $d(N_2(u)-\{u_{31},u_{41}\},\{s_3,s_4\}) \ge g/2-1$. Let $N = G[(C_1 - \{u,u_3,u_4\}) \cup S]$ and let N^* be a copy of N. To derive a contradiction, we construct a smaller $(4,g')$-graph $H = N \cup N^* \cup M$ (see Figure 2), where M is the set of new edges described below:

(a) connect s_i to u_i^* and s_i^* to u_i for $i = 1,2$;

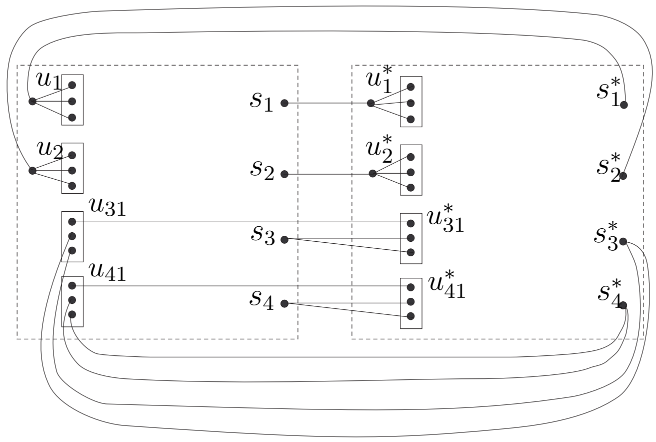

Figure 2: Illustration of the construction in Subcase 2.2 of Lemma 4.

(b) connect s_i to $\{u_{i2}^*, u_{i3}^*\}$ and s_i^* to $\{u_{i2}, u_{i3}\}$ for $i = 3, 4$;

(c) connect u_{i1} to u_{i1}^* for $i = 3, 4$.

Now we calculate the girth of the new graph. If a new cycle $\mathcal{C}$ goes through two edges in (a), then its length is at least $2 + 2(1 + (g/2 - 2)) = g$. If $\mathcal{C}$ goes through two edges in (b), then its length is at least g, since $d(N_2(u) - \{u_{31}, u_{41}\}, \{s_3, s_4\}) \geq g/2 - 1$. If $\mathcal{C}$ goes through two edges in (c), then $\mathcal{C}$ has length at least $2 + 2(g - 4) > g$, since $g \geq 12$. If $\mathcal{C}$ goes through one edge in (a) and one edge in (b), then its length is at least $(g/2 - 2) + (g/2 - 1) + 3 = g$. If it goes through one edge in (c) and one edge in (a) or (b), then taking into account that $g \geq 12$, the length of $\mathcal{C}$ is at least $(g - 4) + (g/2 - 2) + 2 \geq g$ or $(g - 2) + (g/2 - 3) + 2 \geq g$.

Subcase 2.3. The two paths of length $g/2 - 3$ are from $N_2(u)$ to $\{s_1, s_2\}$ and $\{s_3, s_4\}$, respectively.

Without loss of generality, assume that $d(u_{13}, s_1) = g/2 - 3$ and $d(u_{33}, s_3) = g/2 - 3$. Clearly, no paths of length at most $g/2-2$, from $N_2(u)$ to S, can share the same end vertex in $N(s_i) \cap V(C_1)$. Since $|N(s_2) \cap V(C_1)| + |N(s_4) \cap V(C_1)| = 5$, we have $|P_{\leq g/2-2}(N_2(u), S)| \leq 7$. If $|P_{\leq g/2-2}(N_2(u), S)| \leq 6$, then there are four vertex sets of $N_2(u)$ satisfying the conditions in Observation 2 and we thus derive a contradiction. So we assume $|P_{\leq g/2-2}(N_2(u), S)| = 7$. Subsequently there exists a vertex $v \in N(u)$ such that $|P_{\leq g/2-2}(N(v) - u, S)| = 1$ and

$|P_{\leq g/2-2}(N_2(u)-N(v),S)|=6$. Note that $\sum_{i=1}^{4}|N(s_i)\cap V(C_1)|=10$, so $|P_{\leq g/2-2}((N_2(v)-N(u))\cup(N_2(u)-N(v)),S)|\leq 10$. In other words, $|P_{\leq g/2-2}(N_2(v)-N(u),S)|\leq 4$. As mentioned above, $|P_{\leq g/2-2}(N(u)-v,S)|\leq 2$. Hence there are at least six vertices in $N_2(v)$ at distance $g/2-1$ from S. Denote $N(v)=\{u,v_1,v_2,v_3\}$ and $W'_i=N(v_i)-v=\{v_{i1},v_{i2},v_{i3}\}$ $(i=1,2,3)$. If $d(v,S)=g/2-1$, since $|P_{\leq g/2-2}(N(v)-u,S)|=1$ and $\max\{d(v,S)\mid v\in V(C_1)\}=g/2-1$, then there exist two disjoint vertex subsets $T_1\subseteq W'_{j_1}$ and $T_2\subseteq W'_{j_2}$ such that $|T_1|=|T_2|=2$ and $d(T_i,S)\geq g/2-1$, where $i=1,2$, $j_1,j_2\in\{1,2,3\}$. If $d(v,S)=g/2-2$, then $v\in\{u_{13},u_{23}\}$, and there exists a vertex subset $T_1\subseteq W'_j$ of order two such that $d(T_1,S)\geq g/2-1$, for some j. Note that there is also a vertex subset $T_2=\{u_2,u_4\}\subseteq N(u)-v$ of order two such that $d(T_2,S)\geq g/2-1$. Thus, there are in total at least six vertices at distance $g/2-1$ from $N_2(v)$ to S, so it is easy to find two vertices r_1 and r_2 in $N_2(v)-(T_1\cup T_2)$ such that $d(r_1,S)\geq g/2-1$ and $d(r_2,S)\geq g/2-1$. Then G is not a $(4,g)$-cage by Observation 2.

Case 3. $1\leq|P_{g/2-3}(W_i,S)|\leq 2$ *for some* W_i.

Without loss of generality, assume $d(u_1,S)=g/2-2$ and $d(u_j,S)=g/2-1$, for $j=2,3,4$. With Cases 1 and 2 discussed, we can assume that there is exactly one vertex in W_j at distance $g/2-2$ from S, otherwise we can regard u_j as u to show that G is not a $(4,g)$-cage as in Cases 1 or 2. Then we can find two sets T_1, T_2 and two vertices r_1, r_2 in $N_2(u)$ satisfying the conditions of Observation 2, thus G is not a $(4,g)$-cage. □

Lemma 5. *If* $\max\{d(v,S)\mid v\in V(C_1)\}=g/2-1$, $|N(s_i)\cap V(C_2)|=2$ $(i=1,2,3,4)$, *then* G *is not a* $(4,g)$*-cage.*

Proof. Let $d(u,S)=g/2-1$, $N(u)=\{u_1,u_2,u_3,u_4\}$ and $W_i=N(u_i)-u=\{u_{i1},u_{i2},u_{i3}\}$ $(i=1,2,3,4)$.

Case 1. $d(u_i,S)=g/2-2$ $(i=1,2,3,4)$.

Since no paths of length $g/2-3$, from $N_2(u)$ to S, can share the same end vertex in S, there is exactly one vertex, say u_{i3}, in W_i at distance $g/2-3$ from S. Due to the girth requirement, we have $d(W_i-u_{i3},S)\geq g/2-1$. So G is not a $(4,g)$-cage by Observation 2.

Case 2. *There exists a vertex* $v\in N(u)$ *such that* $d(v,S)=g/2-1$.

Suppose $v = u_4$. Let $N(v) = \{v_1, v_2, v_3, u\}$ and $W_i' = N(v_i) - v = \{v_{i1}, v_{i2}, v_{i3}\}$ $(i = 1, 2, 3)$. Since $\sum_{i=1}^{4} |N(s_i) \cap V(C_1)| = 8$, so $|P_{\leq g/2-2}(\cup_{i=1}^{3}(W_i \cup W_i'), S)| \leq 8$. As $\max\{d(v, S) \mid v \in V(C_1)\} = g/2 - 1$, without loss of generality, assume that $d(W_i, S) = d(u_{i3}, S) \leq g/2 - 2$ and $d(W_i', S) = d(v_{i3}, S) \leq g/2 - 2$ $(i = 1, 2, 3)$. Subsequently $|P_{\leq g/2-2}(\cup_{i=1}^{3}(W_i \cup W_i'), S)| \leq 2$.

Subcase 2.1. $|P_{\leq g/2-2}(\cup_{i=1}^{3} W_i, S)| \leq 4$ *and* $|P_{\leq g/2-2}(\cup_{i=1}^{3} W_i', S)| \leq 4$.

We can choose four vertex subsets, say $T_1 \subseteq W_1$, $T_2 \subseteq W_2$, $T_3 \subseteq W_1'$, $T_4 \subseteq W_2'$, such that $|T_i| = 2$ $(i = 1, 2, 3, 4)$ and $d(T_i, S) \geq g/2 - 1$. Without loss of generality, assume $T_1 = \{u_{11}, u_{12}\}$, $T_2 = \{u_{21}, u_{22}\}$, $T_3 = \{v_{11}, v_{12}\}$ and $T_4 = \{v_{21}, v_{22}\}$. By Observation 1, we can assume $d(\{s_1, s_2\}, \{s_3, s_4\}) \geq 3$.

Let $N = G[(C_1 - \{u, v, u_1, u_2, v_1, v_2\}) \cup S]$ and let N^* be a copy of N. Now we construct a 4-regular graph G' (see Figure 3) as follows:

(a) connect s_i to T_i^* and s_i^* to T_i for $i = 1, 2, 3, 4$;

(b) connect u_{i3} to u_{i3}^* and v_{i3} to v_{i3}^* for $i = 1, 2$;

(c) connect u_3 to u_3^* and v_3 to v_3^*.

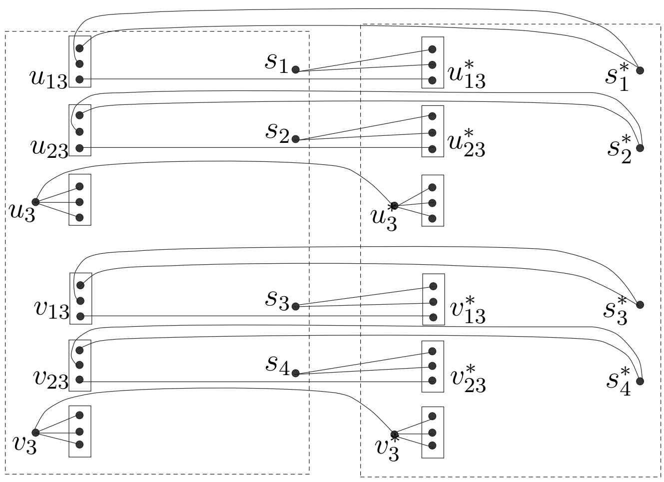

Figure 3: Illustration of the construction in Lemma 5.

In the following, we examine the girth of G'. If the new cycle $\mathcal{C}$ contains two edges both in (a) or (b), then its length is at least g, since $d(\{s_1, s_2\}, \{s_3, s_4\}) \geq 3$. If $\mathcal{C}$ contains two edges both in (c), then it has length at least $2+4+2(g-5) > g$. If $\mathcal{C}$ contains one edge in (a) and another edge in (b), then its length is at least $(g/2-3)+(g-5)+2 \geq g$, since $g \geq 12$. If $\mathcal{C}$ contains one edge in (a) and another edge in (c), then its length is at least $(g/2-3)+(g-5)+4 > g$. If $\mathcal{C}$ contains one edge in (b) and another edge in (c), then its length is at least $2(g-4)+2 > g$. Now G' is a $(4, g')$-graph with girth $g' \geq g$ and smaller order than G, a contradiction.

Subcase 2.2. $|P_{\leq g/2-2}(\cup_{i=1}^{3} W_i, S)| \leq 5$ *or* $|P_{\leq g/2-2}(\cup_{i=1}^{3} W_i', S)| \leq 5$.

Without loss of generality, assume that $|P_{\leq g/2-2}(\cup_{i=1}^{3} W_i, S)| \leq 5$. Then in this case, $|P_{\leq g/2-2}(\cup_{i=1}^{3} W_i', S)| = 3$. Furthermore, there are at least two vertices in W_1 and W_2 at distance less than $g/2-1$ from S, respectively. Otherwise, we can find four vertex subsets as in Subcase 2.1 to obtain a contradiction. $|P_{\leq g/2-2}(\cup_{i=1}^{3}(W_i \cup W_i'), S)| = 8$ and $\sum_{i=1}^{4} |N(s_i) \cap V(C_1)| = 8$. So if $|P_{g/2-3}(\cup_{i=1}^{3} W_i, S)| \geq 2$, then $|P_{g/2-3}(\cup_{i=1}^{3} W_i, S)| = 2$ and $|P_{g/2-3}(\cup_{i=1}^{3} W_i', S)| = 1$. Hence for each vertex $v' \in \cup_{i=1}^{3}(W_i' - v_{i3})$, $d(v', S) = g/2-1$. Assume $d(v_{33}, S) = g/2-3$; then $d(v_1, S) = d(v_2, S) = g/2-1$ and $d(v_{13}, S) = d(v_{23}, S) = g/2-2$. Since $\max\{d(v, S) \mid v \in V(C_1)\} = g/2-1$, each vertex in $\{v_{11}, v_{12}, v_{21}, v_{22}\}$ is at distance $g/2-1$ from S. Due to the girth requirement and $\sum_{i=1}^{4} |N(s_i) \cap V(C_1)| = 8$, besides the shortest paths from each W_i' to S $(i = 1, 2, 3)$, there are at most six other paths of length $g/2-1$, not going through a vertex in $N(v)$, from $\cup_{i=1}^{3} W_i'$ to S. Hence there exists a vertex v_j, $j \in \{1, 2, 3\}$, such that for each vertex $x \in N(v_j) - v$, there is only one path of length less than $g/2$, not going through v_j, from x to S. Then we can find some v_i and four vertex subsets in $N_2(v_i)$ satisfying the conditions of Observation 2, so G is not a $(4, g)$-cage. If $|P_{g/2-3}(\cup_{i=1}^{3} W_i, S)| = 1$, then we can also find four disjoint vertex subsets in $N_2(v)$ satisfying the conditions of Observation 2, since there are two vertices in $\{u_1, u_2, u_3\}$ at distance $g/2-1$ from S and $|P_{\leq g/2-2}(W_i', S)| = 1$ for each W_i'.

The proof is complete. □

Lemma 6. *If* $\max\{d(v, S) \mid v \in V(C_1)\} = g/2$, $|N(s_1) \cap V(C_2)| = |N(s_2) \cap V(C_2)| = 1$ *and* $|N(s_3) \cap V(C_2)| = |N(s_4) \cap V(C_2)| = 2$, *then G is not a $(4, g)$-cage.*

Proof. Let $d(u, S) = g/2$, $N(u) = \{u_1, u_2, u_3, u_4\}$ and $W_i = N(u_i) - u = \{u_{i1}, u_{i2}, u_{i3}\}$

$(i = 1, 2, 3, 4)$.

Claim. There exists a vertex $v \in V(C_1)$ such that $|P_{\leq g/2-2}(N_2(v), S)| \leq 5$.

If the claim is not true, then $|P_{g/2-2}(N_2(u), S)| \geq 6$. If there exists a vertex $u' \in N(u)$ such that $d(u', S) = g/2$, then $|P_{g/2-2}(N_2(u), S)| \leq 5$ or $|P_{g/2-2}(N_2(u'), S)| \leq 5$ as $\sum_{i=1}^{4} |N(s_i) \cap V(C_1)| = 10$. So we may assume $d(u_i, S) = g/2 - 1$ for all neighbors u_i of u. Without loss of generality, assume that $|P_{g/2-2}(W_4, S)| \leq |P_{g/2-2}(W_i, S)|$ $(i = 1, 2, 3)$. Let $m = |P_{g/2-2}(W_4, S)|$. Since $\sum_{i=1}^{4} |N(s_i) \cap V(C_1)| = 10$, so $m \leq 2$. If $m = 2$, then $|P_{g/2-2}(\cup_{i=1}^{3} W_i, S)| \geq 6$, and this yields $|P_{g/2-2}(N_2(u_4), S)| \leq 4$ as $|P_{g/2-2}((N_2(u) - N(u_4)) \cup (N_2(u_4) - N(u)), S)| \leq 10$. If $m = 1$, then $|P_{g/2-2}(\cup_{i=1}^{3} W_i, S)| \geq 5$ as $|P_{g/2-2}(N_2(u), S)| \geq 6$. In other words, $|P_{\leq g/2-2}(N_2(u_4), S)| \leq 5$. Thus the claim is proved.

From the claim it follows that there are at least seven vertices in $N_2(v)$ at distance at least $g/2 - 1$ from S. Then we can find four disjoint vertex subsets of $N_2(v)$ such that $|T_1| = |T_2| = 1$ and $|T_3| = |T_4| = 2$ which satisfy the conditions in Observation 2. Hence G is not a $(4, g)$-cage. □

Lemma 7. *If $\max\{d(v, S) \mid v \in V(C_1)\} = g/2$, $|N(s_i) \cap V(C_2)| = 2$ $(i = 1, 2, 3, 4)$, then G is not a $(4, g)$-cage.*

Proof. Let v be a vertex of $V(C_1)$, $N(v) = \{v_1, v_2, v_3, v_4\}$ and $W_i = N(v_i) - v = \{v_{i1}, v_{i2}, v_{i3}\}$ $(i = 1, 2, 3, 4)$.

Claim. If $|P_{\leq g/2-2}(N_2(v), S)| \leq 4$, then G is not a $(4, g)$-cage.

We may assume that there are at least two vertices in some particular W_i at distance at most $g/2 - 2$ from S, otherwise G is not a $(4, g)$-cage by Observation 2. Consequently there exists a set W_j such that $d(W_j, S) \geq g/2 - 1$. Let $k, l \in \{1, 2, 3, 4\}$. If $d(W_k, s_l) = d(v_{k1}, s_l) = g/2 - 2$, then $d(W_k - v_{k1}, s_l) \geq g/2$. If $d(W_k, s_l) = d(v_{k1}, s_l) = g/2 - 3$, then $d(W_k - v_{k1}, s_l) \geq g/2 + 1$. Let $N = G[(C_1 \cup S) - (N(v) \cup v)]$ and let N^* be a copy of N. Based on the above facts and $|N(s_i) \cap V(C_2)| = 2$ for all i, it is easy to construct a new $(4, g')$-graph (where $g' \geq g$) using N and N^* and with some additional edges. (To illustrate, we show a special case of the construction in Figure 4; the other cases are easy to verify.) Therefore G is not a $(4, g)$-cage by Theorem 1. The claim is proved.

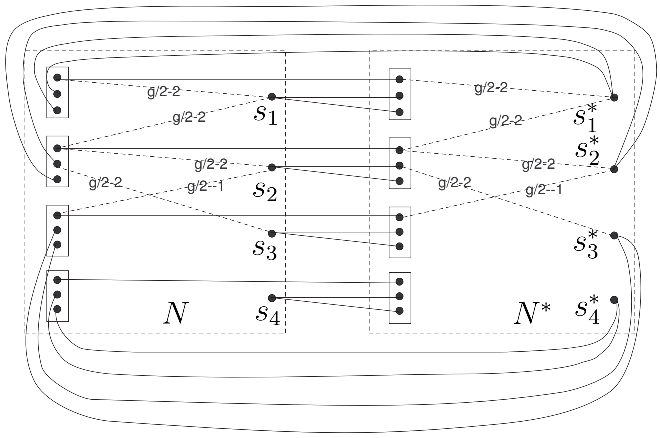

Figure 4: Illustration of the construction in Claim of Lemma 7.

Note that for each edge xy, $|P_{\leq g/2-2}((N_2(x) - N(y)) \cup (N_2(y) - N(x)), S)| \leq 8$ as $\sum_{i=1}^{4} |N(s_i) \cap V(C_1)| = 8$. Let $d(u, S) = g/2$. If there is a vertex $v \in N(u)$ such that $d(v, S) = g/2$, then $|P_{g/2-2}(N_2(u) - N(v), S)| \leq 4$ or $|P_{g/2-2}(N_2(v) - N(u), S)| \leq 4$. Then G is not a $(4, g)$-cage by the claim above. Suppose $d(u_i, S) = g/2 - 1$ for all u_i. Without loss of generality, assume that $|P_{g/2-2}(W_4, S)| \leq |P_{g/2-2}(W_i, S)|$ for $i = 1, 2, 3$. Let $m = |P_{g/2-2}(W_4, S)|$. Since $\sum_{i=1}^{4} |N(s_i) \cap V(C_1)| = 8$, so $m \leq 2$. If $m = 2$, then $|P_{g/2-2}(\cup_{i=1}^{3} W_i, S)| = 6$. Since $\sum_{i=1}^{4} |N(s_i) \cap V(C_1)| = 8$, so $|P_{\leq g/2-2}(N_2(u_4), S)| = 2$. If $m = 1$, then $|P_{\leq g/2-2}(\cup_{i=1}^{3} W_i, S)| \geq 4$. Otherwise for each W_i, $|P_{g/2-2}(W_i, S)| = 1$, then by Observation 2, G is not a $(4, g)$-cage. Hence $|P_{\leq g/2-2}(N_2(u_4), S)| \leq 4$. In both cases, we see that G is not a $(4, g)$-cage by the claim above. □

Lemma 8. *If* $\max\{d(v, S) \mid v \in V(C_1)\} = g/2 + 1$, *then* G *is not a* $(4, g)$*-cage.*

Proof. Suppose $d(u, S) = g/2 + 1$. Then $d(N_2(u), S) \geq g/2 - 1$. It is straightforward to obtain four vertex sets from $N_2(u)$ satisfying the conditions in Observation 2. Hence G is not a $(4, g)$-cage. □

With the preparation above, we are ready to show the main theorem.

Theorem 2. *Every* $(4, g)$*-cage* G *with even girth* $g \geq 12$ *is tightly superconnected.*

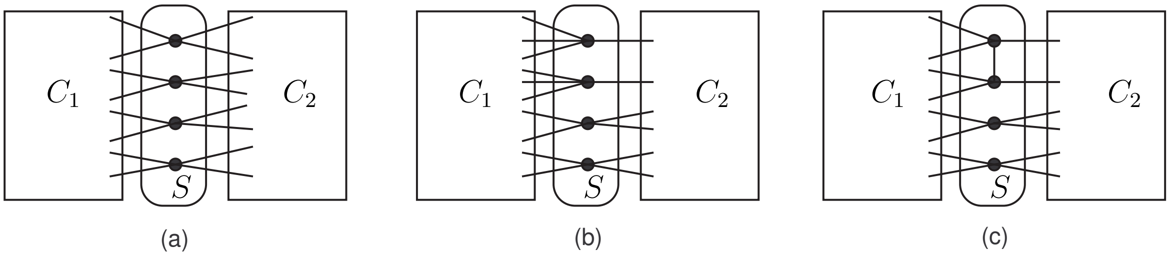

Figure 5: The three cut-sets considered in the proof of Theorem 2.

Proof. Suppose G is not superconnected. Then we choose a non-trivial cut-set S of G such that S minimizes the order of the smaller component C_1 of $G - S$ among all non-trivial cut-sets. Set $C_2 = G - S - C_1$. Since $4|V(C_1)| - E(S, C_1) = \sum_{v \in V(C_1)} d_{C_1}(v) \equiv 0 \pmod 2$, we have $E(S, C_1) \equiv 0 \pmod 2$. Similarly, $E(S, C_2) \equiv 0 \pmod 2$. Since every $(4, g)$-cage is edge-superconnected, we need only discuss the three cases for the cut-sets S shown in Figure 5. Cases (a) and (b) are impossible by Lemmas 4 – 8. For case (c), we can simply delete edge $s_1 s_2$ from $G[S]$ and obtain a contradiction as in Lemmas 5, 7 and 8. So G is superconnected. From Lemma 2, G is tightly superconnected. □

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