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Maximum fractional factors in graphs[☆]

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Abstract

We prove that fractional k-factors can be transformed among themselves by using a new adjusting operation repeatedly. We introduce, analogous to Berge's augmenting path method in matching theory, the technique of increasing walk and derive a characterization of maximum fractional k-factors in graphs. As applications of this characterization, several results about connected fractional 1-factors are obtained.

Keywords: Fractional k-factor; Connected fractional factor; Increasing walk; 1-factor; Toughness

1. Introduction

We study the fractional factor problem in graphs, which can be considered as a relaxation of the well-known cardinality matching problem. The fractional factor problem has wide-ranging applications in areas such as network design, scheduling and the combinatorial polyhedron. For instance, in a communication network if we allow several large data packets to be sent to various destinations through several channels, the efficiency of the network will be improved if we allow the large data packets to be partitioned into small parcels. The feasible assignment of data packets can be seen as a fractional flow problem and it becomes a fractional matching problem when the destinations and sources of a network are disjoint (i.e., the underlying graph is bipartite).

In this work, we consider undirected finite simple graphs only. Denote a graph with vertex set $V(G)$ and edge set $E(G)$ by $G = (V(G), E(G))$. Let x be an end vertex of an edge e; we denote the incidence relation between x and e by $x \sim e$ or $e \sim x$.

Let $f : E(G) \to [0, 1]$ be a real-valued function from the edge set $E(G)$ to the real number interval [0, 1]. For any $e \in E(G)$, $f(e)$ is referred to as the **weight** of the edge e. Define $E_f = \{e \in E(G) : f(e) > 0\}$ and let $G[E_f]$ be the subgraph of G induced by E_f. If $\sum_{e\sim v} f(e) = k$ is satisfied for each vertex $v \in V(G)$, then $G[E_f]$, or G_f for

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short, is called a **fractional k-factor** of G with **indicator function** f. A fractional 1-factor is also called a fractional perfect matching in [7]. On restricting $f : E(G) \to \{0, 1\}$, fractional k-factors and fractional 1-factors are precisely the conventional k-factors and perfect matchings.

A **walk** W is a sequence of edges $e_1, e_2, \ldots, e_m$ so that each pair of any two consecutive edges are incident and each edge appears *no more than twice*. Sometimes, for convenience, we also use the vertices to represent a walk, e.g., $W = (x_1, x_2, \ldots, x_m, x_{m+1})$ where $e_i = x_i x_{i+1}$, $1 \le i \le m$. A **closed walk** is a walk with $x_1 = x_{m+1}$. A **path** is a walk such that all vertices are distinct and a **cycle** is a *closed* path.

Let G_f and G_g be two fractional k-factors of G with indicator functions f and g, respectively. Define $E^+ = \{e \in E(G) : f(e) - g(e) > 0\}$ and $E^- = \{e \in E(G) : f(e) - g(e) < 0\}$. A **sign-alternating walk** of G with respect to f and g is a closed walk of even length with its edges alternately in E^+ and in E^-.

Given a sign-alternating walk of G, one of the major techniques in this work is to adjust weights of the edges on a walk so that either $|E^+|$ or $|E^-|$ decreases. This process is referred as an **adjusting operation** and the details of this process are described below:

Let $C = (e_1, e_2, \ldots, e_{2m})$ be a sign-alternating walk with respect to f and g and $w = f - g$. If an edge e is used twice, we use two parallel edges e' and e'' to replace e and assign the weight $\frac{1}{2}w(e)$ to e' and e'', respectively. Set

$$\varepsilon = \min_{e \in E(C)} \{|w(e)|\}.$$

Without loss of generality, assume that $w(e_1) > 0$. Define a new indicator function f_1: $f_1(e_i) = f(e_i) - \varepsilon$ for i odd, $f_1(e_i) = f(e_i) + \varepsilon$ for i even and $f_1(e) = f(e)$ for any $e \in E(G) - E(C)$.

After the adjusting operation, collapse the parallel edges e' and e'' back to e and let $f_1(e) = f_1(e') + f_1(e'')$. Now G_{f_1} is a fractional k-factor of G with indicator function f_1. Obviously, the cardinality of $E_{f_1 - g}$ is smaller than that of E_{f-g}.

Applying the adjusting operation, we can obtain the following result which is a generalization of the transformation theorem proven in [4].

Theorem 1.1. *Let G_f and G_g be two fractional k-factors of G with indicator functions f and g, respectively. Then G_g can be obtained from G_f through performing finitely many adjusting operations repeatedly.*

Let G_f be a fractional k-factor of G with indicator function f. If $|E_f| \ge |E_g|$ holds for any fractional k-factor G_g with indicator function g, then G_f is called a **maximum** fractional k-factor (i.e., with the maximum number of non-zero edges).

An **increasing walk** W of a graph G with respect to f is a *closed* walk of even length so that the weights of its edges are greater than zero or less than one *alternately* and with at least one edge of *weight zero*. So if we denote an increasing walk by a sequence of edges, $W = (e_0, e_1, \ldots, e_{2m-1}, e_0)$, then $f(e_{2r}) < 1$ and $f(e_{2r+1}) > 0$ for $0 \le r \le m - 1$ and $f(e_{2i}) = 0$ for some i. One example of increasing walks is the graph consisting of two odd cycles joined by a single edge e, where $f(e) = 0$ and the weights of all edges on two odd cycles are positive but less than one. It is easy to see that no two incident edges of the weight zero can appear in an increasing walk.

Using Theorem 1.1 we characterize maximum fractional k-factors as follows.

Theorem 1.2. *Let G_f be a fractional k-factor of a graph G with indicator function f. Then G_f is a maximum fractional k-factor of G if and only if G has no increasing walk with respect to f.*

In fact, Theorem 1.2 is a fractional version of the well-known Berge theorem regarding maximum matchings [1]. One may use Theorem 1.2 to design an efficient algorithm to find a maximum fractional k-factor starting with an arbitrary fractional k-factor.

A **connected fractional factor** of G is a fractional factor G_f if G_f is connected. Finding sufficient conditions for the existence of connected factors has attracted much attention recently (see [2] and [3]). The authors are not aware of any research done for connected fractional factors.

On the basis of Theorem 1.2 we obtain a toughness condition for the existence of connected fractional 1-factors.

For $S \subseteq V(G)$, let $\omega(G - S)$ denote the number of components of $G - S$. The **toughness** of G is defined by

$$t(G) = \begin{cases} +\infty & \text{if } G \text{ is complete;} \\ \min\left\{ \dfrac{|S|}{\omega(G-S)} : \emptyset \subseteq S \subseteq V(G) \text{ and } \omega(G-S) > 1 \right\} & \text{otherwise.} \end{cases}$$

If $t(G) = k$, we say that G is k-**tough**. The concept of toughness was introduced by Chvátal [5] as a new graphic parameter for studying Hamilton cycles in graphs. Toughness has become an effective tool for investigating many other graph theory problems. In particular, Enomoto et al. [6] showed that if a graph G is k-tough, then it has k-factors. Furthermore, we propose the following conjecture.

Conjecture 1. *Let G be a k-tough graph (k is an integer and $k \geq 1$). Then G contains a connected fractional k-factor.*

Using Theorem 1.2, we confirm this conjecture for the case $k = 1$ and obtain the following result.

Theorem 1.3. *Let G be a graph with $t(G) \geq 1$. Then G has a connected fractional 1-factor.*

Note that, unlike that of the usual 1-factor, the existence of the fractional 1-factor may not imply evenness of $|V(G)|$. The toughness condition in Theorem 1.3 is seen to be sharp by noting the graph $G = K_n \bigvee (n+1)K_1$, i.e., the complete join between a complete graph K_n and an independent set $(n+1)K_1$.

We investigate the properties of connected fractional 1-factors in Theorems 1.4 and 1.5.

Theorem 1.4. *Let G_f be a bipartite connected fractional 1-factor of a graph G. Then G_f is 2-connected.*

Theorem 1.5. *Let G be a graph of order at least 5 and let G_f be a connected fractional 1-factor of G with the minimum number of edges. Then G_f contains no 4-cycles.*

The proofs of Theorems 1.1 and 1.2 are presented in Section 2. In Section 3 we prove Theorem 1.3. Finally, we provide the proofs of Theorems 1.4 and 1.5 in Section 4.

2. Proofs of Theorems 1.1 and 1.2

Applying the adjusting operation introduced in Section 1, we here provide a proof of Theorem 1.1.

Proof of Theorem 1.1. Let G_f and G_g be two fractional k-factors of G with indicator functions f and g, respectively. Assume that $f \not\equiv g$. Define an edge set

$$E_{f \neq g} = \{e \in E(G) : f(e) - g(e) \neq 0\}.$$

Then $E_{f \neq g} \neq \emptyset$. Let H be the subgraph of G induced by $E_{f \neq g}$. Assign each edge $e \in E(H)$ the weight $w(e) = f(e) - g(e)$. Define $E_H^+ = E^+ = \{e \in E(H) : w(e) > 0\}$ and $E_H^- = E^- = \{e \in E(H) : w(e) < 0\}$. Let $E_x = \{e \in E(G) : e$ is incident with $x\}$ for a vertex $x \in V(G)$. It is easy to see that for any $x \in V(H)$, $|E_H^+ \cap E_x| \geq 1$ and $|E_H^- \cap E_x| \geq 1$. Thus $\delta(H) \geq 2$.

Next we show the existence of a sign-alternating walk in H with respect to f and g. If there does not exist a sign-alternating cycle in H, we choose a longest sign-alternating path P in H, say $P = (x_1, x_2, \ldots, x_m)$.

Case 1. m is odd.

Without loss of generality, assume $w(x_1x_2) > 0$, then $w(x_{m-1}x_m) < 0$. Since $\delta(H) \geq 2$, there exist $i, j \in \{1, 2, \ldots, m\}$ such that $w(x_1x_i) < 0$ and $w(x_mx_j) > 0$. Thus $C_1 = (x_1, \ldots, x_i, x_1)$ and $C_2 = (x_j, \ldots, x_m, x_j)$ are odd cycles. If $i > j$, then $C = (x_1, \ldots, x_j, x_m, \ldots, x_i, x_1)$ is a sign-alternating cycle of H; we are done. If $i \leq j$, then $C = (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_m, x_j, \ldots, x_i, x_1)$ is a sign-alternating walk of H.

Case 2. m is even.

Like for Case 1, we may assume that $w(x_1x_2) > 0$. Then $w(x_{m-1}x_m) > 0$. Since $\delta(H) \geq 2$, there exist $i, j \in \{1, 2, \ldots, m\}$ such that $w(x_1x_i) < 0$ and $w(x_mx_j) < 0$. Thus cycles $C_1 = (x_1, \ldots, x_i, x_1)$ and $C_2 = (x_j, \ldots, x_m, x_j)$ are odd. If $i > j$, then $(x_1, \ldots, x_j, x_m, \ldots, x_i, x_1)$ is a sign-alternating cycle of H. If $i \leq j$, then $C = (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_m, x_j, \ldots, x_i, x_1)$ is a sign-alternating walk of H.

In both cases we can find a sign-alternating walk in H. Using the adjusting operation introduced in Section 1, we obtain a fractional k-factor of G with indicator function f_1 and $|E_{f_1 - g}| < |E_{f-g}|$. Repeating the adjusting operation on f_1 and g, and so on, thus we obtain a series of fractional k-factors with indicator functions $f = f_0, f_1, \ldots, f_{s-1}, f_s = g$, where $|E_{f_{i+1}-g}| < |E_{f_i - g}|$ for $i = 0, 1, \ldots, s-1$. The proof is complete. $\square$

Proof of Theorem 1.2. Let G_f be a maximum fractional k-factor of G with indicator function f. Suppose that G has an increasing walk $C = (e_1, e_2, \ldots, e_m)$. Let $E'(C) = \{e \in E(C) : 0 < f(e) < 1\}$. Assume that the occurrence of each edge on C is no more than p times. Set

$$\varepsilon = \frac{1}{2p} \min_{e \in E'(C)} \{\min\{f(e), 1 - f(e)\}\}.$$

Without loss of generality, assume that $f(e_1) = 0$. Let $g(e_i) = \varepsilon$ for i odd, $g(e_i) = -\varepsilon$ for i even, and $g(e) = 0$ for $e \in E(G) - E(C)$. Then $f + g$ is still a fractional k-factor of G, but $|E_{f+g}| > |E_f|$, a contradiction.

Conversely, suppose that G has no increasing walk with respect to f. We show that G_f is a maximum fractional k-factor. Otherwise, let G_g be a maximum fractional k-factor with indicator function g and $|E_g| > |E_f|$. Then there exists at least one edge $e_1 \in E(G)$ such that $g(e_1) > 0$ and $f(e_1) = 0$. By Theorem 1.1, we can obtain G_g from G_f through a series of adjusting operations. Let $f = f_0, f_1, \ldots, f_{s-1}, f_s = g$ be the indicator functions generated by the already mentioned adjusting operations and r the smallest subscript such that $f_{r-1}(e_1) = 0$ and $f_r(e_1) > 0$ $(r \geq 1)$. Then in $G_{f_{r-1}}$ there exists a sign-alternating walk C containing e_1, say $C = (e_1, e_2, \ldots, e_m)$. By the definition of sign-alternating walk, it is easy to see that

$$f_{r-1}(e) < f_r(e), \quad \text{if } f(e) - g(e) < 0 \text{ for any } e \in E(C) \tag{1}$$

and

$$f_{r-1}(e) > f_r(e), \quad \text{if } f(e) - g(e) > 0 \text{ for any } e \in E(C). \tag{2}$$

The adjusting operations imply the following inequalities:

$$f_i(e) \leq f_{i+1}(e) \quad \text{for } i = 0, \ldots, s-1, \text{ if } f(e) - g(e) < 0 \text{ for any } e \in E(G) \tag{3}$$

and

$$f_i(e) \geq f_{i+1}(e) \quad \text{for } i = 0, \ldots, s-1, \text{ if } f(e) - g(e) > 0 \text{ for any } e \in E(G). \tag{4}$$

From (1) and (3), we have $f(e_j) \leq f_{r-1}(e_j) < f_r(e_j) \leq g(e_j) \leq 1$, for $j \equiv 1 \pmod 2$. Similarly, from (2) and (4), we obtain $f(e_j) \geq f_{r-1}(e_j) > f_r(e_j) \geq g(e_j) \geq 0$, for $j \equiv 0 \pmod 2$. By the choice of e_1, we see that C is an increasing walk of G with respect to f, a contradiction to the choice of G_f. $\square$

3. Proof of Theorem 1.3

Let G be a graph with the connectivity $\kappa(G)$ and S, T be two disjoint subsets of $V(G)$. Denote the set of edges from S to T by $E_G(S, T)$. To prove Theorem 1.3, we need the following lemmas.

Lemma 3.1 (*See [7]*). *A graph G has a fractional 1-factor if and only if for any $S \subseteq V(G)$,*

$$i(G - S) \leq |S|$$

where $i(G - S)$ denotes the number of isolated vertices in $G - S$.

Lemma 3.2. *Let G be a graph with $t(G) \geq 1$. If $|V(G)| \geq 2$, then G has a fractional 1-factor.*

Proof. If G is complete, obviously G has a fractional 1-factor. Otherwise, suppose that $|V(G)| \geq 2$ and $t(G) \geq 1$ but G has no fractional 1-factors. From Lemma 3.1, there exists a subset S of $V(G)$ such that

$$i(G - S) > |S|.$$

Since G is connected and $S \neq \emptyset$, we have $\omega(G - S) \geq i(G - S) \geq 2$. Therefore

$$t(G) \leq \frac{|S|}{\omega(G - S)} \leq \frac{|S|}{i(G - S)} < 1$$

a contradiction to $t(G) \geq 1$. $\square$

Lemma 3.3 (*See [7]*). *A bipartite graph G has a fractional 1-factor if and only if it has a 1-factor.*

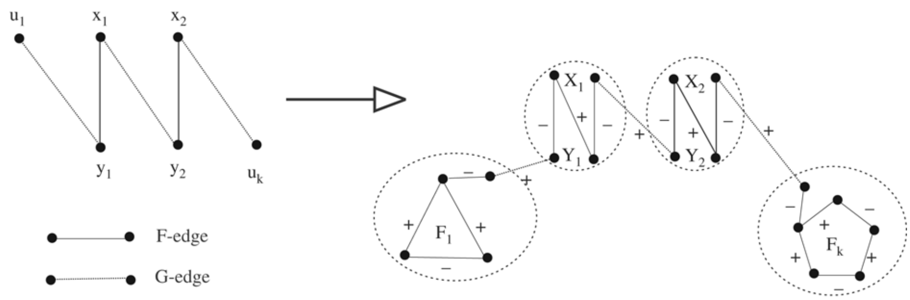

Fig. 1. Illustration of Case 1, where '+' and '−' represent weights greater than zero and less than one, respectively.

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. By Lemma 3.2, G has fractional 1-factors. Suppose that G has no connected fractional 1-factors. Let $F = G_f$ be a maximum fractional 1-factor. By Theorem 1.2, there exists no increasing walk in G with respect to f. Assume that the components of F are $F_1, \ldots, F_m (m \geq 2)$. It is easy to see that if F_i ($\ncong K_2$) is a component of F, then $0 < f(e) < 1$ for each $e \in E(F_i)$ and F_i must contain cycles.

Because $t(G) \geq 1$, we have $\kappa(G) \geq 2$. In particular, G is connected. Denote a bipartite component with bipartition X_i and Y_i by $F_i = (X_i, Y_i)$.

Now we construct a new graph $G' = (V, E) = (V_1 \cup V_2, E_1 \cup E_2 \cup E_3)$, where $V_1 = \bigcup\{x_i, y_i : F_i = (X_i, Y_i) \text{ is bipartite}\}$ and $V_2 = \bigcup\{z_i : F_i \text{ is non-bipartite}\}$, and $E_1 = \bigcup\{x_i y_i : F_i = (X_i, Y_i) \text{ is bipartite}\}$, $E_2 = \bigcup\{u_i v_j : E_G(U_i, V_j) \neq \emptyset, U_i \in \{X_i, Y_i\}, V_j \in \{X_j, Y_j\} \text{ for } i \neq j\}$, where U_i and V_j are the bipartition corresponding to u_i and v_i, and $E_3 = \bigcup\{u_i z_j : E_G(U_i, F_j) \neq \emptyset, U_i \in \{X_i, Y_i\}, F_j \text{ is a non-bipartite component}\}$. The edges in E_1 are called F-edges and the edges in $E_2 \cup E_3$ are called G-edges. Clearly, each F-edge corresponds to an odd path in a bipartite component of G_f and each G-edge corresponds to the edge(s) in $E(G) \setminus E(F)$. An $F - G$ *alternating path* in G' is a path whose edges are alternately F-edges and G-edges. It is not hard to verify that a closed alternating path of even length in G' corresponds to an increasing walk with respect to f in G.

Let $P_{G'} = (u_1, \ldots, u_k)$ be a longest $F - G$ alternating path in G' (i.e., weights of the edges are greater than zero and less then one alternately). Then $P_{G'}$ corresponds to an alternating path P_G with respect to f in G. Moreover, if the end vertex u_1 of $P_{G'}$ is incident to a G-edge, then u_1 is in V_2.

Now we consider three cases.

Case 1. $u_1 \in V_2$ and $u_k \in V_2$.

In this case, since $u_1 \in V_2$ and $u_k \in V_2$, u_1 is connected to an odd cycle C_1 of F_1 by a path P_{u_1} and u_k is connected to an odd cycle C_k of F_k by a path P_{u_k}. Then C_1, P_{u_1}, P_G, P_{u_k} and C_k form an increasing walk in G with respect to f (see Fig. 1), a contradiction.

Case 2. $u_1 \in V_1$ and $u_k \in V_1$.

Without loss of generality, assume $u_1 = x_1$ and $u_k = y_k$. Then either X_1 or Y_k is independent in G. Otherwise, let e_1 be an edge joining two vertices in X_1 and e_k be an edge joining two vertices in Y_k. Let the odd cycle containing e_1 be C_1 and the odd cycle containing e_k be C_k. Then the odd cycles C_1, C_k and the path P_G can form an increasing walk in G with respect to f, a contradiction. Assume that X_1 is independent in G. If a vertex v of X_1 is adjacent to a vertex w outside of the path P_G, then the edge vw corresponds to a G-edge of G' which contradicts the fact that $P_{G'}$ is the longest $F - G$ alternating path. If a vertex v of X_1 is adjacent to a vertex w on the path P_G, then there is a cycle C_1 in G containing the edge vw. If the cycle C_1 is even, then it is an increasing walk in G, a contradiction. If C_1 is odd, then Y_k must be an independent set in G. Applying the same argument as to X_1, we see that no vertex in Y_k is adjacent to a vertex outside P_G and the length of cycle C_k which contains an edge from Y_k to a vertex on P_G is odd. Thus C_1, C_k and an alternating path form an increasing walk in G again, a contradiction. Therefore, the vertices in Y_k are not adjacent to other vertices besides those in X_k. In other words, X_k is a cut set of G. By Lemma 3.3, we have $|X_k| = |Y_k|$. Let $S = X_k$. Then $\omega(G - S) \geq |Y_k| + 1 = |S| + 1$ and thus

$$1 \leq t(G) \leq \frac{|S|}{\omega(G - S)} \leq \frac{|S|}{|S| + 1} < 1,$$

a contradiction.

Case 3. $u_1 \in V_1$ and $u_k \in V_2$.

Without loss of generality, assume $u_1 = x_1$. The arguments in this case are similar to those for Case 2. First, we show that X_1 is independent in G. Otherwise, let e_1 be an edge joining two vertices in X_1 and let an odd cycle C_k in F_k be connected by a path P_k to the path P_G. Then the odd cycle C_1, the paths P_G, P_k and C_k can form an increasing walk in G with respect to f, a contradiction. If a vertex v of X_1 is adjacent to a vertex w outside of the vertices of the path P_G, then the edge vw corresponds to a G-edge of G' which contradicts the fact that $P_{G'}$ is the longest $F - G$ alternating path. Next, if a vertex v of X_1 is adjacent to a vertex w on the path P_G, then it creates a cycle C_1 in G with odd length. Otherwise, if the cycle C_1 has even length, then it is an even alternating cycle and thus an increasing walk in G, a contradiction. Then C_1 and an odd cycle in the non-bipartite component F_k together with an alternating path form an increasing walk in G again, a contradiction. Therefore, the vertices in X_1 is not adjacent to other vertices besides those in Y_1 or Y_1 is a cut set of G. By Lemma 3.3, we have $|X_1| = |Y_1|$. Let $S = Y_1$. Then

$$1 \le t(G) \le \frac{|S|}{\omega(G-S)} \le \frac{|S|}{|S|+1} < 1,$$

a contradiction.

The proof is complete. □

4. Proofs of Theorems 1.4 and 1.5

Two of basic properties of fractional 1-factors were given in [7]; one is Lemma 3.3 and the other is the following lemma.

Lemma 4.1 (*See [7]*). *If a non-bipartite graph G has fractional 1-factors, then it has a fractional 1-factor G_f with indicator function f such that $f(e) \in \{0, \frac{1}{2}, 1\}$ for any $e \in E(G)$.*

Lemmas 3.3 and 4.1 imply that if G has a fractional 1-factor, then there exists a fractional 1-factor consisting of K_2's and odd cycles only. In fact, the fractional 1-factor stated in Lemma 3.3 is a fractional 1-factor of G with the minimum number of edges. Due to the simplicity, such fractional factors are particularly interesting and important. In this section we investigate the properties of connected 1-factors with the minimum number of edges.

Proof of Theorem 1.4. Let $G_f = (X, Y)$ be a bipartite connected fractional 1-factor of G. Then $|X| = |Y|$ from Lemma 3.3. Let $v \in X$ be a cut vertex of G_f and $G_i = (X_i, Y_i)$ $(1 \le i \le m)$ be the components of $G_f - v$. Obviously, $m \ge 2$ and $N(v) \cap Y_i \ne \emptyset$ for $i = 1, \ldots, m$, where $N(v)$ denotes the set of neighbors of v in G_f. However, for any G_i,

$$\sum_{e \in E(G_i)} f(e) = \sum_{x \in X_i} \left(\sum_{e \sim x} f(e) \right) = \sum_{x \in X_i} 1 = |X_i|.$$

Similarly,

$$\sum_{e \in E(G_i)} f(e) < \sum_{y \in Y_i} \left(\sum_{e \sim y} f(e) \right) = \sum_{y \in Y_i} 1 = |Y_i|.$$

Thus, $|X_i| \le |Y_i| - 1$. But

$$|X| = \sum_{i=1}^{m} |X_i| + 1 \le \sum_{i=1}^{m} (|Y_i| - 1) + 1 = |Y| - m + 1 < |Y|,$$

a contradiction. Therefore G_f has no cut vertex or G_f is 2-connected. □

Next we prove Theorem 1.5.

Proof of Theorem 1.5. Let G be a graph of order at least 5 and let G_f be a connected fractional 1-factor with the minimum number of edges in G. We prove that G_f has no 4-cycles. Suppose that G_f has a 4-cycle $C = (v_1, v_2, v_3, v_4, v_1)$.

We consider two cases.

Case 1. The cycle C contains at least one chord.

Suppose that $v_1v_3 \in E(G_f)$. Since $f(v_1v_3) > 0$, then at least one of v_2 and v_4 is of degree at least 3 in G_f. If only one of them, say v_4, is of degree at least 3 in G_f, then $\varepsilon = f(v_3v_4) < f(v_1v_2)$. Starting from v_1v_2, we alternately decrease and increase the weights on C by ε and then it results in a connected fractional 1-factor of G with fewer edges (we say that G_f can be *improved through* C from now on), a contradiction to the choice of G_f. Similarly, if both v_2 and v_4 are of degree at least 3 in G_f, then G_f can be improved through C, a contradiction again.

Case 2. The cycle C contains no chord.

There are at least one pair of adjacent vertices on C, say v_1 and v_2, that have neighbors outside C. If v_4 or v_3 is of degree 2 in G_f, say $d_{G_f}(v_4) = 2$, then $f(v_1v_2) < f(v_3v_4)$. Let $f(v_1v_2) = \varepsilon$. Replacing $f(v_2v_3)$ and $f(v_1v_4)$ by $f(v_2v_3) + \varepsilon$ and $f(v_1v_4) + \varepsilon$, and $f(v_1v_2)$ and $f(v_3v_4)$ by 0 and $f(v_1v_4) - \varepsilon$. Then we obtain a connected fractional 1-factor with one edge fewer, a contradiction. Suppose that $d_{G_f}(v_3) \geq 3$ and $d_{G_f}(v_4) \geq 3$. If $f(v_1v_2) \neq f(v_3v_4)$, $f(v_1v_4) \neq f(v_2v_3)$ or $G_f - \{v_1v_2, v_3v_4\}$ or $G_f - \{v_1v_4, v_2v_3\}$ is connected, then G_f can be improved through C. Otherwise, $f(v_1v_2) = f(v_3v_4)$, and $f(v_1v_4) = f(v_2v_3)$ and both $G_f - \{v_1v_2, v_3v_4\}$ and $G_f - \{v_1v_4, v_2v_3\}$ are disconnected. Let G_1 and G_2 be the components of $G_f - \{v_1v_4, v_2v_3\}$ such that v_1v_2 and v_3v_4 are in G_1 and G_2, respectively. If one of the components, say G_1, is bipartite, then v_1v_2 is contained in an even cycle C_1 of G_1. We can alternately increase and decrease a small positive value on the edges of C_1 such that $f(v_1v_2)$ decreases. Then G_f can be improved, a contradiction. So G_1 and G_2 are non-bipartite. Then we can form a closed even walk W by using two odd cycles, from G_1 and G_2 respectively, and a path through v_1v_4 connecting them. Now we can adjust on W to make $f(v_1v_4)$ decrease and then G_f can be improved again. This last contradiction leads to the conclusion of the proof. □

Finally, we present two open problems.

Problem 1. Find a characterization of the connected fractional k-factors with the minimum number of edges.

Problem 2. Is it true that a connected bipartite fractional 1-factor with the minimum number of edges is a minimally 2-connected graph?

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