The Fractional Vertex Arboricity of Graphs

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Abstract

The vertex arboricity $va(G)$ of a graph G is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces an acyclic subgraph. The fractional version of the vertex arboricity is introduced in this paper. We determine the fractional vertex arboricity for several classes of graphs, e.g., complete multipartite graphs, cycles, integer distance graphs, prisms and Peterson graphs.

Keywords vertex arboricity; tree coloring; fractional vertex arboricity; fractional tree coloring

1 Introduction

In this paper, we use $\mathbb{Z}$ to denote the set of all integers and $|S|$ for the cardinality of a set S ($|S| = +\infty$ means that S is an infinite set).

A *k-coloring* of a graph G is a mapping g from $V(G)$ to $\{1, 2, ..., k\}$. With respect to a given k-coloring, V_i denotes the set of all vertices of G colored with i, and $\langle V_i \rangle$ denotes the subgraph induced by V_i in G. If V_i induces a subgraph whose connected components are trees, then g is called a *k-tree coloring*. The *vertex arboricity* of a graph G, denoted by $va(G)$, is the minimum integer k for which G has a k-tree coloring. In other words, the vertex arboricity $va(G)$ of G is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces an acyclic subgraph (i.e., a forest).

In fact, if V_i is an independent set for each i $(1 \le i \le k)$, then f is called a *proper* k-coloring and the chromatic number $\chi(G)$ of a graph G is the minimum integer k of

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colors for which G has a proper k-coloring. So the proper coloring is a special case of the tree coloring.

Kronk *et al.* [4] proved that $va(G) \le \lceil \frac{\Delta(G)+1}{2} \rceil$ for any graph G. Chartrand *et al.* [2] showed $va(K(p_1, p_2, \cdots, p_n)) = n - \max\{k| \sum_0^k p_i \le n-k\}$ for the complete n-partite graph $K(p_1, p_2, \cdots, p_n)$, where $p_0 = 0, 1 \le p_1 \le p_2 \le \cdots \le p_n$.

In this paper, we introduce the fractional version of vertex arboricity and to determine the fractional vertex arboricity for several families of graphs. This is the first paper in a series of investigations on the fractional vertex arboricity, its relationship with other graphic parameters.

2 The fractional vertex arboricity of graphs

Let S be a set system whose elements are subsets of a set V. A *covering* of V is a collection of elements $L_1, L_2, \cdots, L_j$ of S such that $V \subseteq L_1 \cup \cdots \cup L_j$.

For any graph G, let $\mathcal{F}(G)$ be the set of all subsets of $V(G)$ that induce forests of G. Clearly, $\mathcal{F}(G)$ is a set system of $V(G)$.

We can define the fractional vertex arboricity $va_f(G)$ of a graph G as follows.

Definition: A *fractional tree coloring* of a graph G is a mapping g from $\mathcal{F}(G)$ to the interval $[0, 1]$ such that

$$\sum_{L \text{ contains } x} g(L) \ge 1 \qquad \text{for any} \quad x \in V(G).$$

The *weight* of a fractional tree coloring is the sum of its values, and the *fractional vertex arboricity* of a graph G is the minimum possible weight of a fractional coloring, that is,

$$va_f(G) = \min\{\sum_{L \in \mathcal{F}(G)} g(L) \mid g \text{ is a fractional tree coloring of G}\}.$$

Clearly, we have $va_f(H) \le va_f(G)$ for any subgraph $H \subseteq G$.

If we restrict the range of a mapping g to $\{0, 1\}$ instead of $[0, 1]$, then $va_f(G)$ is the usual vertex arboricity, $va(G)$.

If g is a $va(G)$-tree coloring of G and $V_i = \{v \mid v \in V(G),\ f(v) = i\}$ $(1 \le i \le va(G))$, then we can define a mapping h: $\mathcal{F}(G) \longrightarrow [0, 1]$ by

$$h(L) = \begin{cases} 1 & \text{for } L = V_i,\ 1 \le i \le va(G) \\ 0 & \text{otherwise} \end{cases}$$

such that h is a fractional tree coloring of G which has the weight $va(G)$. Therefore, it follows immediately that $va_f(G) \leq va(G)$.

Conversely, if G has a $(0,1)$-valued fractional tree coloring g of weight k. Then the support of g consists of k forests $V_1, V_2, \cdots, V_k$ whose union is $V(G)$. If we color any vertex v with the smallest i such that $v \in V_i$, then we have a k-tree coloring of G. Thus the vertex arboricity of G is the minimum weight of a $(0,1)$-valued fractional tree coloring.

Remark: The vertex arboricity of a finite graph G can be seen as an optimal solution of an integer programming and its fractional version can be viewed as an optimal solution of its relaxed problem, i.e., a linear programming problem.

To each set $L_i \in \mathcal{F}(G)$ we associate a $(0,1)$-variable x_i with it. The vector $\mathbf{x} = \{x_i\}$ is an indicator of the sets we have selected for the covering. Let M be the vertex-forest incident matrix of G, i.e., the $(0,1)$-matrix whose rows are indexed by $V(G)$, columns are indexed by $\mathcal{F}(G)$ and (i,j)-entry is 1 only when $v_i \in L_j$. The condition that the indicator vector $\mathbf{x}$ corresponds to a covering is simply $M\mathbf{x} \geq \mathbf{1}$ (that is, every coordinate of $M\mathbf{x}$ is at least 1). Hence the vertex arboricity of G is precisely the optimal value of the integer programming

$$\begin{array}{ll} Min & \sum_i x_i \\ Subject\ to & \\ & M\mathbf{x} \geq \mathbf{1} \\ & x_i = 0 \ or \ 1 \quad (1 \leq i \leq |\mathcal{F}(G)|) \end{array} \tag{1}$$

The relaxation of the integer programming (1) is the following linear programming

$$\begin{array}{ll} Min & \sum_i x_i \\ Subject\ to & \\ & M\mathbf{x} \geq \mathbf{1} \\ & 0 \leq x_i \leq 1 \quad (1 \leq i \leq |\mathcal{F}(G)|) \end{array} \tag{2}$$

and the optimal value of (2) is the fractional vertex arboricity of G.

Using the well-known relation between the dual problems, we can derive the lower bound for $va_f(G)$.

Lemma 2.1. *Let G be a finite graph, $t = \max\{|L| : L \in \mathcal{F}(G)\}$, then $va_f(G) \geq \frac{|V(G)|}{t}$.*

Proof. The dual linear programming of (2) is the following

$$\begin{array}{ll} Max & \sum_j y_j \\ Subject\ to & \\ & M^T\mathbf{y} \leq \mathbf{1} \\ & 0 \leq y_j \leq 1 \quad (1 \leq i \leq |V|) \end{array} \tag{3}$$

Thus, if we define f to take the value $f(v)$ on each vertex of $V(G)$ with $0 \leq f(v) \leq 1$ and $M^T\mathbf{y} \leq \mathbf{1}$ for $\mathbf{y} = (f(v_1), \cdots, f(v_n))^T$ with $n = |V|$, then $\mathbf{y}$ is a feasible solution of (3).

Let ω be the objective value of (3) for some feasible solution $\mathbf{y}$. Since (2) and (3) are a pair of dual problems, from Weak Duality Theorem (see [1]), we have $\omega \leq va_f(G)$.

If we assign each vertex of G weight $\frac{1}{t}$, then we have a feasible solution of (3). Thus $va_f(G) \geq \frac{|V(G)|}{t}$. □

Therefore, $va_f(G) \geq 1$ for any nonempty graph G. Clearly, $va_f(G) = 1$ if a graph G is a forest.

For a complete $n-$partite graph $G = K(m_1, m_2, \cdots, m_n)$, we denote the vertices of n-partite of $V(G)$ by

$$X_1 = \{v_{11}, v_{12}, \cdots, v_{1m_1}\},\ X_2 = \{v_{21}, v_{22}, \cdots, v_{2m_2}\},\ \cdots,\ X_n = \{v_{n1}, v_{n2}, \cdots, v_{nm_n}\}$$

where $|X_i| = m_i$ for $1 \leq i \leq n$.

Theorem 2.2. *For a complete $n-$partite graph $G = K(m_1, m_2, \cdots, m_n)$,*

$$va_f(G) = \begin{cases} n - \frac{n}{m+1} & \text{for } m_1 = m_2 = \cdots = m_n = m \geq n > 2 \\ \frac{2n}{3} & \text{for } m_1 = m_2 = \cdots = m_n = m = 2 \\ \frac{n}{2} & \text{for } m_1 = m_2 = \cdots = m_n = 1 \end{cases}$$

and

$$n - \frac{m}{m+1} \leq va_f(G) \leq n - \frac{m(n+1)}{(m+1)^2}$$

for $m_1 = m_2 = \cdots = m_{n-1} = m > m_n = n$.

Proof. (1) For $m \geq n$, it is easy to see that $t = \max\{|X| : X \in \mathcal{F}(G)\} = m + 1$. So $va_f(G) \geq \frac{mn}{m+1} = n - \frac{n}{m+1}$ by Lemma 2.1. Define a mapping $h_1 : \mathcal{F}(G) \longrightarrow [0, 1]$ by

$$h_1(X) = \begin{cases} \frac{1}{(m+1)(n-1)} & \text{for } X = X_i \cup \{v_{kj}\},\ 1 \leq i, j, k \leq n, i \neq k, \\ 0 & \text{otherwise.} \end{cases}$$

Since there are exactly $(m+1)(n-1)$ forests that have nonzero weights containing vertex v_{ij} for $1 \leq i, j \leq m$, h_1 is a fractional tree coloring of G. The number of $(m+1)$-forests that contain m elements in X_i is $\binom{n-1}{1}\binom{m}{1} = m(n-1)$. So there are $nm(n-1)$

elements in $\mathcal{F}$ that have nonzero values or $va_f(G) \leq \frac{nm(n-1)}{(m+1)(n-1)} = n - \frac{n}{m+1}$. Therefore $va_f(G) = n - \frac{n}{m+1}$.

(2) For $m = 2$, it is straight forward to verify that $t = \max\{|X| : X \in \mathcal{F}(G)\} = 3$. So $va_f(G) \geq \frac{2n}{3}$. Define a mapping $h_2 : \mathcal{F}(G) \to [0, 1]$ by

$$h_2(X) = \begin{cases} \frac{1}{3(n-1)} & \text{for } |X| = 3 \text{ and there exist } i < j \text{ such that } X \subseteq X_i \cup X_j, \\ 0 & \text{otherwise.} \end{cases}$$

The number of all 3-forests that contain two elements in X_1 is $2(n-1)$ and the number of all 3-forests that contain one element in X_1 is also $2(n-1)$. So there are $4(n-1)+4(n-2)+\cdots+8+4 = 2(n-1)n$ elements in $\mathcal{F}$ that have nonzero values. Then h_2 is a fractional tree coloring of G which has weight $\frac{1}{3(n-1)}2(n-1)n = \frac{2n}{3}$ or $va_f(G) \leq \frac{2n}{3}$. Therefore $va_f(G) = \frac{2n}{3}$.

(3) For $m_1 = m_2 = \cdots = m_n = 1$, define a mapping $h_3 : \mathcal{F}(G) \to [0, 1]$ by

$$h_3(X) = \begin{cases} \frac{1}{n-1} & \text{if } |X| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then h_3 is a fractional tree coloring of G which has weight $\frac{n}{2}$. Thus $va_f(G) \leq \frac{n}{2}$. It is easy to see that $t = \max\{|X| : X \in \mathcal{F}(G)\} = 2$, so $va_f(G) \geq \frac{|V(G)|}{t} = \frac{n}{2}$. Hence, $va_f(G) = \frac{n}{2}$.

(4) For $m_1 = m_2 = \cdots = m_{n-1} = m > n$ and $m_n = n$, define a mapping $h_4 : \mathcal{F}(G) \to [0, 1]$ by

$$h_4(X) = \begin{cases} \frac{1}{(n-1)(m+1)} & \text{if } X = X_i \cup \{v_{nj}\} \text{ for } i < n \text{ or } X = X_n \cup \{v_{kj}\} \text{ for } k < n, \\ \frac{nm-m-2}{(n-1)(m+1)^2(n-2)} & \text{if } X = X_i \cup \{v_{kj}\} \text{ for } i, k < n, \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to verify that h_4 is a fractional tree coloring. Moreover, there are $n(n-1)+(n-1)m$ forests that contain elements of V_n and have nonzero values, $\binom{n-1}{1}\binom{n-2}{1}\binom{m}{1}$ forests that do not contain any element of V_n and have nonzero values. Hence, h_4 has the weight

$$\frac{n+m}{m+1}+(n-1)(n-2)m\frac{nm-m-2}{(n-1)(m+1)^2(n-2)} = \frac{n+m}{m+1}+m\frac{nm-m-2}{(m+1)^2} = n-\frac{m(n+1)}{(m+1)^2}.$$

So $va_f(G) \leq n - \frac{m(n+1)}{(m+1)^2}$.

Since $t = \max\{|X| : X \in \mathcal{F}(G)\} = m + 1$, so $va_f(G) \geq \frac{|V(G)|}{t} = \frac{n+(n-1)m}{m+1} = n - 1 + \frac{1}{m+1} = n - \frac{m}{m+1}$. □

Next, we determine the fractional vertex arboricities of several familiar graphs: cycles, prism of cycles and the Petersen graph.

Theorem 2.3. (1) *For an n-cycle C_n, $va_f(C_n) = \frac{n}{n-1}$.*

(2) *Let L_h be the prism of two h-cycles ($h \geq 3$). Then $\frac{2h}{h+1} \leq va_f(L_h) \leq 2$.*

(3) *For the Petersen graph $P(5,2)$, we have $va_f(P(5,2)) = \frac{10}{7}$.*

Proof. (1) Suppose that $C_n = a_0a_1 \cdots a_{n-1}a_0$. Let $P_i = a_ia_{i+1} \cdots a_{i+n-2}$, where the subscripts are taken with modulo n and $0 \leq i \leq n-1$. It is obvious that every a_i is contained in exactly $n-1$ paths $P_0, \cdots, P_i, P_{i+2}, \cdots, P_{n-1}$. Define a mapping $g : \mathcal{F} \rightarrow [0,1]$ by

$$g(X) = \begin{cases} \frac{1}{n-1} & \text{if } X = P_i, i = 0, 1, \cdots, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Then g is a fractional tree coloring of C_n which has weight $\Sigma_{X \in \mathcal{F}(C_n)} g(X) = \frac{n}{n-1}$, so $va_f(C_n) \leq \frac{n}{n-1}$. Clearly, $t = \max\{|X| : X \in \mathcal{F}(C_n)\} = n-1$, hence $va_f(C_n) \geq \frac{n}{n-1}$. Therefore $va_f(C_n) = \frac{n}{n-1}$.

(2) Denote the $2h$ vertices of the prism L_h by $u_1, u_2, \cdots, u_h$ and $v_1, v_2, \cdots, v_h$. Then the edges of L_h are u_iu_{i+1}, v_iv_{i+1} and u_iv_i $(1 \leq i \leq n)$. Clearly, $t = max\{|X| \mid X \in \mathcal{F}\} = h+1$ and thus $va_f(L_h) \geq \frac{2h}{h+1}$. If we color the vertices $u_1, u_2, \cdots, u_{h-1}, v_{h-1}, v_1$ by 0 and the vertices $v_2, v_3, \cdots, v_{h-2}, v_h, u_h$ by 1, then this is a tree coloring. Thus $va_f(L_h) \leq va(L_h) \leq 2$.

(3) Denote the vertex set of the Petersen graph $P(5,2)$ by $\{a, b, c, d, e, a_1, b_1, c_1, d_1, e_1\}$ and the edge set by $\{ab, bc, cd, de, ea, aa_1, bb_1, cc_1, dd_1, ee_1, a_1c_1, a_1d_1, b_1d_1, b_1e_1, c_1e_1\}$. Since any eight vertices of $P(5,2)$ would induce a cycle, we see that $\max\{|X| : X \in \mathcal{F}\} = 7$. Then $va_f(P(5,2)) \geq \frac{10}{7}$ by Lemma 2.1.

Let

$$\begin{array}{ll} S_1 = \{a, b, c, d, a_1, b_1, e_1\}, & S_2 = \{a, b, c, d, d_1, c_1, e_1\}, \\ S_3 = \{b, c, d, e, e_1, a_1, d_1\}, & S_4 = \{b_1, b, c, d, e, c_1, a_1\}, \\ S_5 = \{c_1, c, d, e, a, d_1, b_1\}, & S_6 = \{c, d, e, a, a_1, e_1, b_1\}, \\ S_7 = \{d, d_1, e, a, b, e_1, c_1\}, & S_8 = \{d, e, a, b, b_1, a_1, c_1\}, \\ S_9 = \{e, a, b, c, c_1, b_1, d_1\}, & S_{10} = \{e_1, e, a, b, c, a_1, d_1\}, \\ S_{11} = \{a, a_1, c_1, c, d, e_1, b_1\}, & S_{12} = \{a, a_1, d_1, d, c, b_1, e_1\}, \\ S_{13} = \{b, b_1, d_1, d, e, a_1, c_1\}, & S_{14} = \{b, b_1, e_1, e, d, c_1, a_1\}, \\ S_{15} = \{c, c_1, e_1, e, a, b_1, d_1\}, & S_{16} = \{c, c_1, a_1, a, e, b_1, d_1\}, \\ S_{17} = \{d, d_1, b_1, b, a, c_1, e_1\}, & S_{18} = \{d, d_1, a_1, a, b, c_1, e_1\}, \\ S_{19} = \{e, e_1, b_1, b, c, d_1, a_1\}, & S_{20} = \{e, e_1, c_1, c, b, a_1, d_1\}. \end{array}$$

Clearly, each S_i $(1 \leq i \leq 20)$ induces a forest and each vertex is contained in exactly fourteen such forests. Define a mapping g by

$$g(X) = \begin{cases} \frac{1}{14} & \text{if } X = S_i, 1 \leq i \leq 20, \\ 0 & \text{otherwise,} \end{cases}$$

then g is a fractional tree coloring which has weight $\frac{20}{14} = \frac{10}{7}$. Hence, $va_f(P(5,2)) \leq \frac{10}{7}$ and then $va_f(P(5,2)) = \frac{10}{7}$. $\square$

In general, it is rather difficult to determine the exact values of either the vertex arboricity or the fractional vertex arboricity for *infinite* graphs. In the following, we investigate a family of special infinite graphs, integer distance graphs and are able to determine the values of the fractional vertex arboricity of some cases. For a set D of positive integers, the *integer distance graph* is a graph with the vertex set $\mathbb{Z}$ and two vertices x and y are adjacent if and only if $|x-y| \in D$, where D is called the *distance set*.

Theorem 2.4. (1) *For $D = \{1, 2, \cdots, m\}$, the fractional vertex arboricity of the integer distance graph $G(D)$ is $\frac{m+1}{2}$, i.e., $va_f(G(D)) = \frac{m+1}{2}$.*

(2) *Let P be the set of all prime numbers, then $va_f(G(P)) = 2$.*

Proof. (1) Let

$$\begin{aligned}
&S_0 = \{\cdots, 0, 1, m+1, m+2, 2(m+1), 2(m+1)+1, \cdots\},\\
&S_1 = \{\cdots, 1, 2, m+2, m+3, 2(m+1)+1, 2(m+1)+2, \cdots\},\\
&S_2 = \{\cdots, 2, 3, m+3, m+4, 2(m+1)+2, 2(m+1)+3, \cdots\},\\
&\vdots\\
&S_{m-1} = \{\cdots, -2, -1, m-1, m, 2m, 2m+1, 3m+1, 3m+2, \cdots\},\\
&S_m = \{\cdots, -1, 0, m, m+1, 2m+1, 2m+2, 2(m+1)+m, 3(m+1), \cdots\}.
\end{aligned}$$

Then each of $S_0, S_1, \cdots, S_m$ induces a forest and each integer i is contained in exactly two S_j $(0 \leq j \leq m)$. Define a mapping $g : \mathcal{F} \to [0,1]$ by

$$g(X) = \begin{cases} \frac{1}{2} & \text{if } X = S_j, j = 0, 1, \cdots, m, \\ 0 & \text{otherwise.} \end{cases}$$

Then g is a fractional tree coloring of $G(D)$ which has the weight $\Sigma_{X\in\mathcal{F}(G(D))} g(X) = \frac{m+1}{2}$, so $va_f(G(D)) \leq \frac{m+1}{2}$.

Let H be a subgraph induced by vertices $0, 1, \cdots, m$. Then H is a complete graph of order $m+1$ and thus that $va_f(G(D)) \geq va_f(H) = \frac{m+1}{2}$ by Theorem 2.2. Therefore, $va_f(G(D)) = \frac{m+1}{2}$.

(2) Let $S_i = \{n \mid n \equiv i \pmod 2, n \in \mathbb{Z}\}$ $(i = 0, 1)$, then S_i induces a forest. It is obvious that each integer is contained in exactly one of such forests. Define a mapping $g : \mathcal{F} \to [0,1]$ by

$$g(X) = \begin{cases} 1 & \text{if } X = S_i, i = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then g is a fractional tree coloring which has the weight 2. So $va_f(G(P)) \leq 2$. Suppose that H is the subgraph induced by vertices $0, 1, 2, \cdots, 7$. It is easy to verify that $t = \max\{|X| : X \subseteq V(H) \text{ and X induces a forest of H}\} = 4$ and the vertex subset $\{0, 1, 2, 3\}$ induces a tree. So $va_f(H) \geq \frac{8}{4} = 2$. Hence $va_f(G(P)) = 2$. □

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