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RESEARCH ARTICLE

On vertex-coloring 13-edge-weighting*

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Abstract L. Addario-Berry et al. [Discrete Appl. Math., 2008, 156: 1168–1174] have shown that there exists a 16-edge-weighting such that the induced vertex coloring is proper. In this note, we improve their result and prove that there exists a 13-edge-weighting of a graph G, such that its induced vertex coloring of G is proper. This result is one step close to the original conjecture posed by M. Karoński et al. [J. Combin. Theory, Ser. B, 2004, 91: 151–157].

Keywords Vertex coloring, edge weighting, degree constrained subgraph
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1 Introduction

All graphs considered are simple. We use $E(S,T)$ to denote the set of edges with one end in S and the other in T. If v is an end vertex of edge e, we write it $e \sim v$. A *k-edge-weighting* of a graph G is an assignment of an integer weight $w(e) \in \{1, 2, \ldots, k\}$ to each edge $e \in E(G)$. An edge weighting naturally induces a vertex coloring w by defining

$$w(u) = \sum_{e \sim u} w(e)$$

for every $u \in V(G)$. We refer this coloring as an *induced coloring*. A k-edge-weighting is *vertex coloring* if the induced coloring w is proper, i.e., $w(u) \neq w(v)$ for any edge $uv \in E(G)$. In Ref. [3], Karoński, Luczak and Thomason initiated the study of vertex coloring edge weighting. Clearly, a graph with a component isomorphic to K_2 cannot have a vertex coloring edge

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weighting. They made the following conjecture.

Conjecture Every graph without an edge component admits a vertex coloring 3-edge-weighting.

There are several partial results towards to this conjecture. Karoński et al. [3] verified this conjecture for 3-colorable graphs. Chang et al.[1)] showed that all the trees and the regular bipartite graphs have vertex coloring 2-edge-weighting. For general graphs, Addario-Berry et al. [1] proved that every graph without an edge component permits a vertex coloring 30-edge-weighting. Recently, they improved the required edge-weighting to 16.

Theorem 1 [2] *Every graph without an edge component permits a vertex coloring* 16*-edge-weighting.*

In this note, base on the technique developed in Ref. [2] but with some refinements, we are able to reduces the required weighting to 13.

Theorem 2 *Every graph without an edge component permits a vertex coloring* 13*-edge-weighting.*

2 Preliminary results

Before proving the main theorem, we need some preliminary results. The degree constrained subgraphs play crucial roles in the proof of the main theorem, so let us start with a degree constrained lemma, which is proved by Addario-Berry et al. in Ref. [2].

Lemma 1 *Given a bipartite graph G with bipartition X and Y. For each $v \in X$, let $a_v^- = \lfloor d(v)/2 \rfloor$ and $a_v^+ = a_v^- + 1$. For each $v \in Y$, choose arbitrary integers a_v^-, a_v^+ satisfying $0 \leqslant a_v^- \leqslant d(v)/2 \leqslant a_v^+$ and*

$$a_v^+ \leqslant \min\left\{\frac{d(v)+a_v^-}{2}+1,\ 2a_v^- + 1\right\}. \tag{2.1}$$

Then there exists a spanning subgraph F of G, such that $d_F(v) \in \{a_v^-, a_v^+\}$ for all $v \in V$.

Remark 1 There is a minor glitch in the proof of Lemma 1 in Ref. [2]. But, with a slight adjustment in the statement (i.e., $a_v^- \leqslant d(v)/2 \leqslant a_v^+$), the original proof of the main result in Ref. [2] can be carried out as it is.

The following lemma is implied in the proof of Theorem 1 in Ref. [3], we use it several times in our proof.

Lemma 2 *Given a connected non-bipartite graph $G = (V, E)$, a set of target colors c_v for all $v \in V$, and a positive integer k, where k is odd or $\sum_{v\in V} c_v$ is even, there exists a k-edge-weighting of G such that for all $v \in V$,*

$$\sum_{e\sim v} w(e) \equiv c_v \pmod{k}.$$

1) Chang G J, Lu C, Wu J, Yu Q. Vertex coloring 2-edge weighting of bipartite graphs

3 Proof of main result

Proof of Theorem 2 Obviously, we only have to consider the connected graph.

If G is bipartite, it has been proven in Ref. [3] that there exists a vertex coloring 3-edge-weighting. So we may assume that G is non-bipartite.

Let G be a nonempty graph (i.e., has at least one edge) and an ordered pair (V_1, V_1') be a partition of the vertices of G, so that the number of edges between V_1 and V_1' is maximized over all the ordered partitions, moreover, V_1 is minimized with respect to the maximum. Such an ordered pair (V_1, V_1') is called a *maximum* 2*-cut* of G.

Firstly, we investigate the properties of maximum 2-cuts. Note that if G is nonempty, then there must exist a maximum 2-cut (V_1, V_1'), and V_1 is a nonempty proper subset of V. Clearly, by the minimality of V_1, all the isolated vertices of G belong to V_1'. Let (V_1, V_1') be a maximum 2-cut of G, and v be a arbitrary vertex in V_1 with degree d in $G[V_1]$. Then there exists at least $d+1$ edges in $E(v, V_1')$. We call these edges the *forward edges* of v, and the d neighbors of v in V_1 the *backward neighbors* of v.

Let (V_1, V_1') be a maximum 2-cut of G, and L be a collection of the bipartite components of $G[V_1']$. Let $R = G - V_1 - L$. If R is a nonempty graph, then we have a maximum 2-cut (V_2, V_2') of R. If $G[V_2']$ is a nonempty graph, then we can find a maximum 2-cut (V_3, V_3') of $G[V_2']$, and so on, to generate V_4, and let $V_5 = V_4'$.

Assume that V_5 exists, in other words, R, $G[V_2']$ and $G[V_3']$ are all nonempty graph. If, in a certain step, the graph become empty before reaching V_5, then we stop, and use the similar argument present below to obtain a vertex coloring 12-edge-weighting of G (in this case, we have a stronger result). So we may assume that $V_5 \neq \emptyset$.

Let

$$Y = \{u \in V_5 \mid u \text{ is not isolated in } V_5\}.$$

From the construction, every vertex u in Y has at least $8d_{G[V_5]}(u) \geqslant 8$ edges joining to V_1. We choose a subset E_u with $8d_{G[V_5]}(u)$ such edges. Let B be the bipartite graph induced by $\cup_{u \in Y} E_u$. If $v \in V_1$ is adjacent to an even (resp. odd) number of edges in B, then place v into the set V_1^{e} (resp. V_1^{o}). Also, partition the vertex-set of L into two sets L_a and L_b based on a 2-coloring of L.

Next, define a vertex coloring c_v on V:

$$c_v = \begin{cases} 0, & v \in V_1^{\mathrm{e}}, \\ 2, & v \in V_1^{\mathrm{o}}, \\ 1, & v \in L_a, \\ 3, & v \in L_b, \end{cases}$$

We assign c_v for other vertices v in R such that $\sum_{v \in V} c_v$ is even (see

Table 1). By Lemma 2, there exists a 4-edge-weighting w of G such that

$$\sum_{e\sim v} w(e) \equiv c_v \pmod 4.$$

Then we discard the weights of edges with one end in R, that is, we only need the weights of edges in $G[V_1 \cup L]$.

Table 1 Value of c_v

V_1^{e}	V_1^{o}	L_a	L_b
0	2	1	3

Process the vertices of V_1 in arbitrary order. For any vertex $v \in V_1$, if all the edges incident with v are weighted, i.e.,

$$N_G(v) \cap V_1' \subseteq L,$$

then we may add 4 to one edge e_v in $E(v, L)$ to adjust the induced coloring of v, such that its coloring is $0 \pmod 8$. (Clearly, $v \in V_1^{\mathrm{e}}$. Note that

$$0 \neq |E(v, L)| \geqslant \frac{1}{2}\, d_G(v),$$

so such an edge must exist.) Otherwise, we assign weight 3 to each unweighted forward edge of v. Now, if the induced coloring of v is not as specified in Table 2, then we can add a weight between 1 and 7 to an edge $e_v \in E(v, R)$ to adjust the induced coloring of v so that it is as specified in Table 2. Denote the new induced coloring of v by w_v'. If v has d backward neighbors in V_1, then it has at least $d+1$ forward edges. For any edge in $E(v, V_1') \setminus \{e_v\}$, we can add 8 to its edge weight. Therefore, we have $d+1$ values in $W_v = \{w_v', w_v'+8, \ldots, w_v'+8d\}$ as specified in Table 2. If u is a processed backward neighbor of v with current coloring w_u, we say that u *blocks* the range $[w_u - 2, w_u + 2]$. Since each processed backward neighbor blocks a range of size 4, so it can block at most one value in W_v. Thus we have at least one value, say w_v, in W_v that is not blocked by any processed backward neighbors of v. Hence we can always add 8 to some edges in $E(v, V_1') \setminus \{e_v\}$ so that the induced coloring of v equals to w_v which is not blocked by any processed backward neighbors.

Table 2 Induced coloring of v under mod 8

V_1^{e}	V_1^{o}	L_a	L_b
0	2	1 or 5	3 or 7

After processing the vertices in V_1, all edges with one end in $V_1 \cup L$ have weights between 1 and 12. Moreover, the edges with one end in V_1 and the other in R have weights between 3 and 11. For any vertices in R, let c_v' be the sum of weights on the weighted edges that are incident with v; if all incident edges of v are unweighted, let $c'_v = 0$. Put nonnegative integer c''_v for every vertex in R so that $c'_v + c''_v$ is as specified in Table 3. Note that every component of R is non-bipartite and every component has at least one vertex in V_2. Since every vertex in V_2 can only have

$$c'_v + c''_v \equiv 1 \text{ or } 2 \pmod 4,$$

so we can choose c''_v so that the sum of c''_v for every component of R is even. By Lemma 2, we have a 4-edge-weighting w'' of R such that

$$\sum_{E(R)\ni e\sim v} w''(e) \equiv c''_v \pmod 4$$

for each vertex $v \in V(R)$. Before this step, all the edges of R are unweighted. Now, all the edges are weighted. Clearly, the induced colorings of vertices in R are as specified in Table 3. Next, process the vertices in the order V_2, V_3, V_4. If the induced coloring of v is not as specified in Table 4, then we can add weight 4 to one forward edge e_v of v, so that its coloring is as specified in Table 4. In our construction, all isolated vertices are put into the second vertex set of the maximum 2-cut, thus such a forward edge must exist. After processing the vertices in V_2, V_3, V_4, the induced coloring for every vertex in R is as specified in Table 4. As in V_1, we can add 8 to some forward edges so that the induced coloring of v is different from the coloring of backward neighbors. Now, denote the induced coloring of a vertex v in G by w_v.

Table 3 Induced coloring of a vertex v in R under mod 4

V_2	V_3	V_4	V_5
1 or 2	1	2	3

Table 4 Final induced coloring of a vertex $v \in V$ under mod 8

V_1	L_a	L_b	V_2	V_3	V_4	V_5
0 or 4	1 or 5	3 or 7	1 or 2	5	6	3 or 7

The remaining work to do includes: adjusting weights in B, finding the desired subgraph F in Lemma 1 and verifying validity of (2.1). The arguments for this are very similar to those in Ref. [2], but for completeness, we include the details here for the readers.

Now, we need to adjust the weights of the edges in B to distinguish colorings of adjacent vertices in V_5 and ensure that the induced colorings of all vertices in V_1 are either 0 or 4 (mod 8), while preventing any new conflict in V_1. In this process, the degree constrained subgraphs in Lemma 1 play crucial roles. Let F be a bipartite subgraph determined by (X, Y), where

$$X = V_1 \cap V(B), \quad Y = V_5 \cap V(B).$$

For each edge $e \in E(F)$, we add 2 to its weight, and for each $e \in E(B-F)$, we subtract 2.

Our goal is to find the required subgraph F. Choose $\{a_v^-, a_v^+\}$ for each vertex in X as follows: for each $v \in X$, let $a_v^- = \lfloor d_B(v)/2 \rfloor$ and $a_v^+ = a_v^- + 1$. Choose $\{a_v^-, a_v^+\}$ for each vertex in Y as follows: process the vertices of Y in arbitrary order, for each $v \in Y$ in turn, we choose $a_v^- \in [d_B(v)/4, d_B(v)/2]$ (recall that 8 divides $d_B(v)$, so this range has integer endpoints), and set

$$a_v^+ = a_v^- + \frac{d_B(v)}{4} + 1.$$

In this process, we make our choice of $\{a_v^-, a_v^+\}$ to ensure that for any previously processed neighbor $u \in Y$, any $a_v \in \{a_v^-, a_v^+\}$ and any $a_u \in \{a_u^-, a_u^+\}$,

$$w_v + 2a_v - 2(d_B(v) - a_v) \neq w_u + 2a_u - 2(d_B(u) - a_u)$$

holds. Define

$$f_v(x) = w_v + 2x - 2(d_B(v) - x)$$

for each vertex $v \in Y$. For distinct integers $x, y \in [d_B(v)/4, d_B(v)/2]$, the pairs

$$\Big\{f_v(x),\, f_v\Big(x + \frac{d_B(v)}{4} + 1\Big)\Big\}, \quad \Big\{f_v(y),\, f_v\Big(y + \frac{d_B(v)}{4} + 1\Big)\Big\}$$

are disjoint. Then for any processed neighbor u, the pair $\{f_u(a_u^-), f_u(a_u^+)\}$ can intersect at most two pairs of choices for $\{a_v^-, a_v^+\}$, but there are precisely $2d_{G[V_5]}(v) + 1$ choices for $\{a_v^-, a_v^+\}$, so the choice for $\{a_v^-, a_v^+\}$ that we need must exist.

Finally, to verify that the chosen $\{a_v^-, a_v^+\}$ satisfies the conditions of Lemma 1. For $v \in X$, the degree choice is exactly the same. For $v \in Y$, we have

$$a_v^- \leqslant \frac{d_B(v)}{2} \leqslant a_v^+.$$

To see (2.1) holds: for $v \in Y$, since

$$a_v^- \leqslant \frac{d_B(v)}{2},$$

we have

$$\begin{aligned} a_v^+ &= a_v^- + \frac{d_B(v)}{4} + 1 \\ &= \frac{d_B(v)}{4} + \frac{a_v^-}{2} + \frac{a_v^-}{2} + 1 \\ &\leqslant \frac{d_B(v)}{2} + \frac{a_v^-}{2} + 1; \end{aligned}$$

from

$$a_v^- \geqslant \frac{d_B(v)}{4},$$

we have

$$a_v^+ = a_v^- + \frac{d_B(v)}{4} + 1 \leqslant 2a_v^- + 1,$$

that is, (2.1) holds for any $v \in Y$. Thus, by Lemma 1, there exists a subgraph F in B such that after performing the additions/subtractions described as above, all adjacent vertices in V_5 have different colorings. Furthermore, the induced coloring of all vertices in V_5 are either 3 or 7 (mod 8).

The induced colorings of vertices in V_1^{e} either stay the same or increase by 4, and thus are now either 0 or 4 (mod 8). Moreover, no conflicts are created within V_1^{e}, because colorings of adjacent vertices were initially at least 8 apart before performing the additions/subtractions in B. Similarly, the induced colorings of vertices in V_1^{o} are now either 0 or 4 (mod 8), and there are no conflicts created within V_1^{o}. Let $uv \in E(G)$ with $u \in V_1^{\mathrm{e}}$ and $v \in V_1^{\mathrm{o}}$. If u and v have the same coloring, then by a simple counting,

$$|w_u - w_v| = 2,$$

a contradiction to the fact that u and v are at least 3 apart by the blocking property. So no conflicts are created within V_1.

Thus, we have achieved the target arises from Table 4. It is easy to see that every edge end up with a weight in the range $[1, 13]$, and the induced colorings on the vertices of V form a proper coloring of G. □

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