Component Factors with Large Components in Graphs

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Abstract

In this paper we obtain sufficient conditions using isolated vertices for component factors with each component of order at least three. In particular, we show that if a graph G satisfies $iso(G-S) \le |S|/2$ for all $S \subset V(G)$, then G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor, where $iso(G-S)$ denotes the number of isolated vertices in $G-S$.

1 Introduction

In this paper we consider component factors of graphs, which are defined as follows. For a set $\mathcal{S}$ of connected graphs, a spanning subgraph F of a graph G is called an $\mathcal{S}$-*factor* of G if every component of F is an element of $\mathcal{S}$. An $\mathcal{S}$-factor is also referred as a *component factor*. There have been many papers on component factors of graphs, but in most cases, $\mathcal{S}$ contains K_2 (i.e., a single edge), but it is relatively rare that $\mathcal{S}$ contains no small component. In addition, it is known that if $\mathcal{S}$ does not contain K_2, then in most cases finding a criterion for a graph to have an $\mathcal{S}$-factor is very difficult since finding a maximum $\mathcal{S}$-subgraph of a given graph is an NP-complete problem. In this paper we obtain several sufficient conditions in terms of the number of isolated vertices for a graph to have a component factor such that each component has order at least three.

We begin with some notation and definitions. We consider a finite simple graph G with vertex set $V(G)$ and edge set $E(G)$, which has neither loops nor multiple edges. We denote by $|G|$ the order of G. For a subset $S \subseteq V(G)$, $G-S$ denotes the subgraph of G induced by $V(G)-S$. For a vertex v of G, the degree of v and the neighborhood of v in G are denoted by $d_G(v)$ and $N_G(v)$, respectively. In particular, $d_G(v) = |N_G(v)|$. The minimum

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degree and the maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Denote by $\alpha(G)$ the independence number of G, which is the maximum cardinality among the independent sets of vertices of G. Let $iso(G)$ and $Iso(G)$ denote the number of isolated vertices and the set of isolated vertices of G, respectively. In particular, $iso(G) = |Iso(G)|$. For sets X and Y, $X \subset Y$ means that X is a proper subset of Y.

We denote the complete graph, the path and the cycle of order n by K_n, P_n and C_n, respectively. We denote the complete bipartite graph by $K_{n,m}$. A criterion for a graph to have a star-factor is given below.

Theorem 1. (Amahashi and Kano [1]) *A graph G has a star-factor, i.e., $\{K_{1,1}, \ldots, K_{1,n}\}$-factor, if and only if $iso(G - S) \leq n|S|$ for all $S \subset V(G)$.*

A graph R is called *factor-critical* if for every vertex x of R, $R - x$ has a 1-factor (K_2-factor). A graph H is called a *sun* if $H = K_1$, $H = K_2$ or H is the corona of a factor-critical graph R with order at least three, i.e., H is obtained from R by adding a new vertex $w = w(v)$ together with a new edge vw for every vertex v of R (Figure 1). A sun with order at least 6 is called a *big sun*. The number of sum components of G is denoted by $sun(G)$. The next theorem gives a criterion for a graph to have a path-factor each of whose components is of order at least three. Note that a shorter proof of the following theorem and a formula for a maximum $\{P_3, P_4, P_5\}$-subgraph of a graph was given in [3].

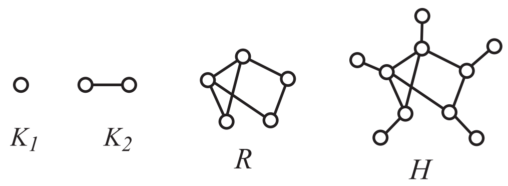

Figure 1: A factor-critical graph R and the sun H obtained from R.

Theorem 2. (Kaneko [2]) *A graph G has a $\{P_3, P_4, P_5\}$-factor (i.e., $P_{\geq 3}$-factor) if and only if $sun(G - S) \leq 2|S|$ for all $S \subset V(G)$.*

In this paper we consider the following problem, and give partial answers to the problem.

Problem 1. *Let G be a graph and λ be a positive rational number. If $iso(G - S) \leq \lambda|S|$ for all $\emptyset \neq S \subset V(G)$, what factor does G have?*

2 Component Factors with Large Components

In this section, we first prove the next theorem.

Theorem 3. *If a graph G satisfies*

$$iso(G - S) \leq \frac{2}{3}|S| \qquad \textit{for all} \quad S \subset V(G),$$

then G has a $\{P_3, P_4, P_5\}$-factor.

Proof. Suppose that G satisfies the condition but has no $\{P_3, P_4, P_5\}$-factor. By Theorem 2, there exists a subset $S \subset V(G)$ such that $sun(G-S) > 2|S|$. Assume that there exist a isolated vertices, b K_2's and c big sun components $H_1, H_2, \ldots, H_c$, where $|H_i| \geq 6$, in $G-S$. We choose one vertex from each K_2 component of $G-S$, and denote the set of such vertices by X. Then $|X| = b$. For each H_i, let R_i denote the factor-critical subgraph of H_i and let $Y_i = V(R_i)$. Then $iso(H_i - Y_i) = |Y_i| = |H_i|/2$. Let $Y = \cup_{i=1}^{r} Y_i$. So we have

$$iso(G - (S \cup X \cup Y)) = a + b + \sum_{i=1}^{c} \frac{|H_i|}{2}.$$

Moreover, it follows that

$$\begin{aligned}
|S \cup X \cup Y| &< \frac{sun(G-S)}{2} + |X| + |Y| \qquad \text{(from } sun(G-S) > 2|S|\text{)} \\
&= \frac{a+b+c}{2} + b + \sum_{i=1}^{c} \frac{|H_i|}{2} \\
&\leq \frac{3}{2}\left(a + b + \sum_{i=1}^{c} \frac{|H_i|}{2}\right) = \frac{3}{2} iso(G - (S \cup X \cup Y)).
\end{aligned}$$

This contradicts the condition that $iso(G - S') \leq (2/3)|S'|$ for all $S' \subset V(G)$. ■

Let $m \geq 1$ be an integer Let $G = K_m + (2m+1)K_2$, which is a graph obtained from K_m and $(2m+1)K_2$ by joining every vertex of K_m to every vertex of $(2m+1)K_2$. Then G has no $\{P_3, P_4, P_5\}$-factor. Let $T \subseteq V(G)$ be an independent set with $|T| \geq 2$. Then $T \subseteq V((2m+1)K_2)$ and so $|N_G(T)| = |T| + m$. If $|T| \leq 2m$, then $i(G - N_G(T)) \leq 2|N_G(T)|/3$, otherwise $i(G - N_G(T)) = 2|N_G(T)|/3 + 1 = 2m + 1$. Since $\delta(G) \geq m + 1 \geq 2$, so $i(G - S) \leq 2|S|/3 + 1$ for all $S \subseteq V(G)$. Therefore the condition of Theorem 3 is sharp.

The next lemma is knows as Harlem Theorem, which is a generalization of Hall's Theorem.

Lemma 1. *Let G be a bipartite graph with bipartition (U, W), and $f : U \to \{1, 2, 3, \ldots\}$. If $|W| = \sum_{x \in U} f(x)$ and*

$$|N_G(S)| \geq \sum_{x \in S} f(x) \qquad \textit{for all} \quad \emptyset \neq S \subseteq U,$$

then G has a star-factor F such that each vertex u of U satisfies $d_F(u) = f(u)$, that is, every u is the center of a star $K_{1,f(u)}$ in F.

We next consider graphs satisfying $iso(G - S) \leq |S|/2$ for all $S \subset V(G)$.

Lemma 2. *If $|G| \leq 6$ and $iso(G - S) \leq |S|/2$ for all $S \subset V(G)$, then G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.*

Proof. It is clear that if G satisfies the condition, then $\delta(G) \geq 2$ and $|G| \geq 3$. If $|G| = 3$, then G is connected and has a $K_{1,2}$-factor. If $|G| = 4$, then $\Delta(G) = 3$, which implies that G has a $K_{1,3}$-factor. Assume $|G| = 5$. If G has two non-adjacent vertices x and y, then $2 = |\{x, y\}| = iso(G - (V(G) - \{x, y\})) \leq |V(G) - \{x, y\}|/2 = 3/2$, a contradiction. Hence G is a complete graph K_5, and so it has a K_5-factor. Now we consider the case of $|G| = 6$. By Theorem 2, G has a $\{P_3, P_4, P_5\}$-factor, say F. Then F must be a P_3-factor, which is a $K_{1,2}$-factor. Therefore the lemma holds. ■

Theorem 4. *If a graph G satisfies*

$$iso(G - S) \leq \frac{|S|}{2} \qquad \text{for all} \quad S \subseteq V(G),$$

then G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

Proof. It is clear that $|G| \geq 3$ and $\delta(G) \geq 2$. Use induction on the lexicographic order of $(|G|, |E(G)|)$. So we assume that the theorem holds for a graph H with either $|H| < |G|$ or $|H| = |G|$ and $|E(H)| < |E(G)|$. Moreover, we may assume that G is connected and $|G| \geq 7$ by Lemma 2. Let

$$\beta = \min \left\{ \frac{|S|}{2} - iso(G - S) \;\middle|\; S \subset V(G) \text{ and } iso(G - S) \geq 1 \right\}.$$

Then $\beta \geq 0$ as $iso(G - S) \leq |S|/2$. For a vertex x with $d_G(x) = \delta(G)$, we have $\beta \leq |N_G(x)|/2 - iso(G - N_G(x))$ and so

$$\delta(G) = d_G(x) = |N_G(x)| \geq 2(\beta + iso(G - N_G(x))) \geq 2(\beta + 1). \tag{1}$$

Take a maximal vertex subset S such that $|S|/2 - iso(G - S) = \beta$. Then

$$\frac{|S'|}{2} - iso(G - S') > \beta \qquad \text{for all} \qquad S \subset S' \subset V(G). \tag{2}$$

Claim 1. $G - S$ has no component of order two or three.

Assume that $G - S$ has a component D isomorphic to K_2. Let $V(D) = \{x, y\}$. Then

$$\begin{aligned} & \frac{|S \cup \{x\}|}{2} - iso(G - (S \cup \{x\})) \\ = \; & \frac{|S| + 1}{2} - (iso(G - S) + 1) < \beta, \end{aligned}$$

a contradiction.

Assume that $G - S$ has a component D of order three. Let $V(D) = \{x, y, z\}$. Then

$$\begin{aligned} & \frac{|S \cup \{x, y\}|}{2} - iso(G - (S \cup \{x, y\})) \\ = \; & \frac{|S| + 2}{2} - (iso(G - S) + 1) = \beta, \end{aligned}$$

a contradiction to the maximality of S.

Claim 2. Every component D of $G - S$ with $|D| \geq 4$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

Let X be a non-empty subset of $V(D)$. Then by (2), we have

$$\frac{|S \cup X|}{2} - iso(G - (S \cup X)) > \beta = \frac{|S|}{2} - iso(G - S).$$

Thus $|X|/2 > iso(D - X)$, which implies that D has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor by the induction hypothesis.

By Claim 1, let $G - S = aK_1 \cup \big(D_1 \cup \cdots \cup D_c\big)$, where $V(aK_1) = Iso(G - S) = \{u_1, \ldots, u_a\}$ and each D_i is a component of $G - S$ with $|D_i| \geq 4$. It is immediate that

$$a = iso(G - S) = |S|/2 - \beta \geq 1. \tag{3}$$

We construct a bipartite graph B with vertex set $V(B) = S \cup U$, where $U = \{u_1, u_2, \ldots, u_a\}$, such that two vertices $u_i \in U$ and $x \in S$ are adjacent in B if and only if u_i and x are joined by an edge of G.

Claim 3. For every $\emptyset \neq Y \subseteq U$, we have $|N_B(Y)| \geq 2|Y| + 2\beta$, and $|N_B(U)| = 2|U| + 2\beta = |S|$.

It follows from (3) and the choice of S that $|N_B(U)| = |S| = 2a + 2\beta = 2|U| + 2\beta$. Assume that there exists a subset $\emptyset \neq Y' \subset U$ such that $N_B(Y') < 2|Y'| + 2\beta$. Then, by the definition of β, $N_B(Y') = N_G(Y') \subset S$ satisfies

$$|Y'| \leq iso(G - N_G(Y')) \leq \frac{|N_G(Y')|}{2} - \beta < |Y'|,$$

a contradiction. Hence the claim holds.

Claim 4. If $\beta \geq 2$, then the theorem holds.

Assume $\beta \geq 2$. Then $\delta(G) \geq 6$ by (1). It is obvious that G has an edge e such that $G - e$ is connected. Let $X \subset V(G - e) = V(G)$. If $iso(G - X) \geq 1$, then

$$iso(G - e - X) \leq iso(G - X) + 2 \leq \frac{|X|}{2} - \beta + 2 \leq \frac{|X|}{2}.$$

If $iso(G - X) = 0$, then $iso(G - e - X) \leq 2$. Further $iso(G - e - X) \geq 1$ implies $|X| \geq 5$ as $\delta(G - e) \geq 5$. Hence if $iso(G - X) = 0$, then $iso(G - e - X) \leq 2 \leq |X|/2$. Therefore by the induction hypothesis, $G - e$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor, which is of course the desired factor of G.

From Claim 4 and the definition of β, it remains to consider the cases of $\beta \in \{0, 1/2, 1, 3/2\}$. Note that $|S| = 2|U| + 2\beta$.

Case 1. $\beta = 0$.

Define $f : U \to \{1, 2, 3 \ldots\}$ by $f(u) = 2$ for all $u \in U$. Then by Lemma 1 and Claim 3, B has a $K_{1,2}$-factor with centers in U. Hence by Claim 2, G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

Case 2. $\beta = 1/2$.

In this case, $|S| = 2|U| + 1$. Choose a vertex $u_1 \in U$ and define $f : U \to \{1, 2, 3, \ldots\}$ by $f(u_1) = 3$ and $f(u_i) = 2$ for all $u_i \in U - \{u_1\}$. Then $|N_B(Y)| \geq \sum_{x \in Y} f(x)$ for all $Y \subseteq U$ by Claim 3. Hence by Lemma 1, B has a $\{K_{1,2}, K_{1,3}\}$-factor. Therefore we can obtain a $\{K_{1,2}, K_{1,3}, K_5\}$-factor of G.

Case 3. $\beta = 1$.

Clearly, $\delta(G) \geq 4$ by (1). We consider two subcases.

Subcase 3.1. $|U| \geq 2$.

In this case, $|S| = 2|U| + 2$. Choose two vertex $u_1, u_2 \in U$ and define $f : U \to \{1, 2, 3, \ldots\}$ by $f(u_1) = f(u_2) = 3$ and $f(u_i) = 2$ for all $u_i \in U - \{u_1, u_2\}$. Then $|N_B(Y)| \geq \sum_{x \in Y} f(x)$ for all $Y \subseteq U$ by Claim 3. Hence, by Lemma 1, B has a $\{K_{1,2}, K_{1,3}\}$-factor and so G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

Subcase 3.2. $|U| = iso(G - S) = 1$.

In this case, $|S| = 2|U| + 2 = 4$ and $V(G) \neq S \cup U$. Let $U = \{u\}$ and $S = \{s_1, s_2, s_3, s_4\}$. If $S \cup \{u\}$ induces a complete graph K_5 in G, then G has the desired $\{K_{1,2}, K_{1,3}, K_5\}$-factor by Claims 1 and 2. So $S \cup \{u\}$ does not induce a complete graph K_5. Without loss of generality, we may assume that s_3 and s_4 are not adjacent in G.

Considering $G - \{s_1, u, s_2\}$, if $iso(G - \{s_1, u, s_2\} - X) \leq |X|/2$ for all $X \subseteq V(G) - \{s_1, u, s_2\}$, then the result is followed by induction hypothesis. So we may assume that there exists $\emptyset \neq R \subseteq V(G) - \{s_1, u, s_2\}$ such that $iso(G - \{s_1, u, s_2\} - R) \geq (|R| + 1)/2$. We choose maximal such a vertex subset R. Then Claims 1 and 2 hold for $G - \{s_1, u, s_2\} - R$ by the maximality of R. Moreover,

$$\frac{|R \cup \{s_1, u, s_2\}|}{2} - iso(G - \{s_1, u, s_2\} - R) \leq \frac{|R| + 3}{2} - \frac{|R| + 1}{2} = 1.$$

Since $\beta = 1$, we obtain

$$\frac{|R \cup \{s_1, u, s_2\}|}{2} - iso(G - \{s_1, u, s_2\} - R) = 1.$$

Therefore $|R|$ is odd. If $|R| \geq 3$, then $S' = R \cup \{s_1, u, s_2\}$ satisfies $|S'|/2 - iso(G - S') = \beta = 1$ and $iso(G - S') \geq 2$. So the result is followed with the similar discussion as in Subcase 3.1.

So we assume $|R| = 1$ and thus $iso(G - \{s_1, u, s_2\} - R) = 1$. Let $R = \{r\}$ and $Iso(G - \{s_1, u, s_2\} - r) = \{y\}$. Since $\delta(G) \geq 4$, we have $d_G(y) = 4$ and $N_G(y) = \{u, s_1, s_2, r\}$. Recall that $N_G(u) = \{s_1, s_2, s_3, s_4\} = S$, so $y \in S$, say $y = s_3$.

If $r \in S$ (i.e., $r = s_4$), then $yr = s_3 s_4$ is an edge of G, which contradicts the fact that s_3 and s_4 are not adjacent in G. Hence $r \notin S$. Let $M = G - (S \cup \{u, r\})$. Then for every $\emptyset \neq Y \subset V(M)$, it follows from (2) and $\{u\} = Iso(G - (S \cup Y \cup \{r\}) - Iso(M - Y)$ that

$$iso(M - Y) = iso(G - (S \cup Y \cup \{r\}) - 1 < \frac{|S| + |Y| + 1}{2} - \beta - 1 = \frac{|Y| + 1}{2}.$$

Hence $iso(M - Y) \leq |Y|/2$, and so by induction, M has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor, and this factor can be extended to a $\{K_{1,2}, K_{1,3}, K_5\}$-factor of G by adding two $K_{1,2}$'s with centres u and y.

Case 4. $\beta = 3/2$.

By (1), we have $\delta(G) \geq 5$. Let $uv, vw \in E(G)$. Then for every $X \subseteq V(G) - \{u, v, w\}$ with $iso(G - \{u, v, w\} - X) \geq 1$, it follows that

$$iso(G - \{u, v, w\} - X) \leq \frac{|X \cup \{u, v, w\}|}{2} - \beta \leq \frac{|X|}{2}.$$

If $iso(G - \{u, v, w\} - X) = 0$, then obviously $iso(G - \{u, v, w\} - X) \leq |X|/2$. Hence by the induction hypothesis, $G - \{u, v, w\}$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor, which can be extended to a $\{K_{1,2}, K_{1,3}, K_5\}$-factor of G.

Consequently the theorem is proved. ■

We now show that the condition in Theorem 4 is sharp. Consider a graph G given in Figure 2. Then G satisfies $iso(G - S) \leq (|S| + 1)/2$ for all $S \subset V(G)$, but has no $\{K_{1,2}, K_{1,3}, K_5\}$-factor. Hence the condition of the theorem is sharp in this sense. The condition of Theorem 4 is sufficient but not necessary. For example, let $G = K_{1,3}$ (or C_{3m}, where $m \geq 2$). Then G contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor but dissatisfies the condition of Theorem 4.

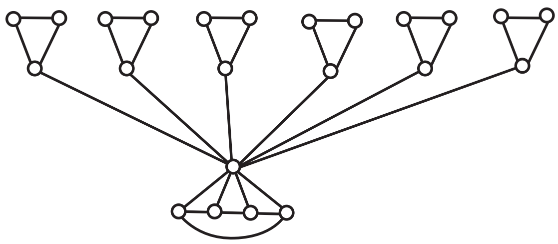

Figure 2: A graph has no $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

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