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General fractional f-factor numbers of graphs

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ABSTRACT

Let G be a graph and f an integer-valued function on $V(G)$. Let h be a function that assigns each edge to a number in [0, 1], such that the *f-fractional number* of G is the supremum of $\sum_{e\in E(G)} h(e)$ over all fractional functions h satisfying $\sum_{e\sim v} h(e) \le f(v)$ for every vertex $v \in V(G)$. An *f-fractional factor* is a spanning subgraph such that $\sum_{v\sim e} h(e) = f(v)$ for every vertex v. In this work, we provide a new formula for computing the fractional numbers by using Lovász's Structure Theorem. This formula generalizes the formula given in [Y. Liu, G. Z. Liu, The fractional matching numbers of graphs, Networks 40 (2002) 228–231] for the fractional matching numbers.

1. Introduction

All graphs considered in this work will be simple finite undirected graphs. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We use $d_G(x)$ for the degree of a vertex x in G. For any $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. We write $G - S$ for $G[V(G) - S]$. We refer the reader to [1] for standard graph theoretic terms not defined here.

Let f and g be two nonnegative integer-valued functions on $V(G)$ such that $g(x) \le f(x)$ for every vertex $x \in V(G)$. A spanning subgraph F of G is a *(g, f)-factor* if $g(v) \le d_F(v) \le f(v)$ for all $v \in V(G)$. If $f \equiv g$, then a (g, f)-factor is also called as an *f-factor*. In 1970, Lovász [2] gave a canonical decomposition of $V(G)$ according to its (g, f)-optimal subgraphs. In this work, we only consider $g \equiv f$.

Define $f(S) = \sum_{x\in S} f(x)$. Let $def(G)$ be the *deficiency* of G with respect to an integer-valued function f and be defined as

$$def(G) = \min_{H\subseteq G} \left\{ \sum_{x\in V(H)} |f(x) - d_H(x)| \right\},$$

where H is a spanning subgraph of G. A subgraph H of G is called *f-optimal* if $def(G) = def(H)$. Let M be an f-optimal subgraph; we call $|E(M)|$ the *f-factor number* and denote it by $\mu(G)$. In particular, if $f \equiv 1$, it is usually referred to as the *matching number*. Let $A(G), B(G), C(G), D(G)$ be defined as in Lovász's Structure Theorem (refer to [1]) and for simplicity, denote them by A, B, C, D, respectively. Then

$$C(G) = \{v \in V(G) \mid d_H(v) = f(v) \text{ for every } f\text{-optimal subgraph } H\},$$
$$A(G) = \{v \in V(G) - C(G) \mid d_H(v) \ge f(v) \text{ for every } f\text{-optimal subgraph } H\},$$

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$$B(G) = \{v \in V(G) - C(G) \mid d_H(v) \le f(v) \text{ for every } f\text{-optimal subgraph } H\},$$
$$D(G) = V(G) - A(G) - B(G) - C(G).$$

Let $G[D] = D_1 \cup \cdots \cup D_\tau$. Let M be a subgraph such that $d_M(v) \le f(v)$ for all $v \in A(G)$. For a component D_i of $G[D]$, we refer to D_i as *M-full* if either M contains an edge of $E(D_i, A(G))$ or M misses an edge of $E(V(D_i), B)$; otherwise, D_i is *M-near full*. For any f-optimal subgraph M of G, where $d_M(v) \le f(v)$ for all $v \in A(G)$, the number of nontrivial components of $G[D]$ which are M-near full is denoted by $nc(M)$. Let $nc(G) = \max\{nc(M)\}$, where the maximum is taken over all f-optimal graph M of G with $d_M(v) \le f(v)$ for all $v \in A(G)$. We describe a graph H as *f-critical* if H contains no f-factors, but for any fixed vertex x of $V(H)$, there exists a subgraph K of H such that $d_K(x) = f(x) \pm 1$ and $d_K(y) = f(y)$ for any vertex y $(y \ne x)$.

Let h be a function defined on $E(G)$ such that $h(e) \in [0, 1]$ for every $e \in E(G)$ and the *f-fractional number* of G is the supremum of $\sum_{e \in E(G)} h(e)$ over all fractional functions h satisfying $\sum_{e \sim v} h(e) \le f(v)$ for each v. We denote the f-fractional number by $\mu_f(G)$. Let $E_v = \{e \mid e \sim v\}$ and $E^h = \{e \mid e \in E(G) \text{ and } h(e) \ne 0\}$. We call h a fractional *f-indicator function* of G if $h(E_v) = f(v)$ for each $v \in V(G)$. If H is a spanning subgraph of G such that $E(H) = E^h$, then H is called a *fractional f-factor* of G with indicator function h, or simply a fractional f-factor. Let $def_f(G)$ be the deficiency of the fractional f-factor of G, and be defined as

$$def_f(G) = \min\left\{ \sum_{v \in V(G)} \left(\left| \sum_{e \in E_v} h(e) - f(v) \right| \right) \mid h \text{ is a function defined on } E(G) \text{ such that} \right.$$
$$\left. h(e) \in [0, 1] \text{ for every } e \in E(G) \right\}.$$

In this work, we investigate the relationship between the f-factor number and the fractional f-factor number, provide a new formula for computing the fractional numbers by using Lovász's Structure Theorem, and generalize the formula for the fractional matching number given in [3]. By the definition, clearly

$$\mu_f(G) = \frac{1}{2}(f(V(G)) - def_f(G)).$$

Let f_S be a function on $V(G-S)$ such that $f_S(v) = f(v) - |E(v, S)|$ and f_S^T is the restriction of f_S on the subgraph T of $G-S$. Given an arbitrary f-optimal subgraph F, let $def_F(T)$ denote the deficiency of subset $V(T)$ with respect to the f-factors.

Theorem 1.1 (*Lovász's Structure Theorem*). *Let $D(G)$, $A(G)$, $B(G)$ and $C(G)$ be defined as above. Let F be an f-optimal subgraph. Then*

(i) *every component D_i of $G[D]$ is $f_B^{D_i}$-critical;*
(ii) *for every component D_i of $G[D]$, $def_F(D_i) \le 1$;*
(iii) *$d_F(v) \in \{f(v), f(v) - 1, f(v) + 1\}$ if $v \in D$; $d_F(v) \le f(v)$ if $v \in B$; $d_F(v) \ge f(v)$ if $v \in A$;*
(iv) *$def(G) = def(F) = f(B) + \tau - f(A) - \sum_{v \in B} d_{G-A}(v)$, where τ denotes the number of components of $G[D]$.*

Lu and Yu [4] gave a different interpretation of $A(G), B(G), C(G), D(G)$ by using alternating trails and thus obtained a shorter proof of Lovász's Structure Theorem. Suppose that F is an f-optimal subgraph, where $d_F(v) \le f(v)$ for all $v \in A(G)$, and let $B_0 = \{v \mid d_F(v) < f(v)\}$. An *$M$-alternating trail* is a trail $P = v_0 v_1 \ldots v_k$ with $v_i v_{i+1} \notin F$ for i even and $v_i v_{i+1} \in F$ for i odd. Then we can define A, B, C, D alternatively as follows:

$$D = \{v \mid \exists \text{ both an even and an odd } F\text{-alternating trail from vertices of } B_0 \text{ to } v\},$$
$$B = \{v \mid \exists \text{ an even } F\text{-alternating trail from a vertex of } B_0 \text{ to } v\} - D,$$
$$A = \{v \mid \exists \text{ an odd } F\text{-alternating trail from a vertex of } B_0 \text{ to } v\} - D,$$
$$C = V(G) - A - B - D.$$

With these new notions, more structural properties of f-optimal subgraphs can be obtained.

Theorem 1.2 (*Lu and Yu [4]*). *Let $D(G)$, $A(G)$, $B(G)$ and $C(G)$ be defined as above. Let F be an arbitrary f-optimal subgraph of G. Then*

(i) *for every component D_i of $G[D]$, if $def(D_i) = 0$, then F either contains an edge of $E(D_i, A)$ or misses an edge of $E(D_i, B)$; if $def(D_i) = 1$, then $E(D_i, B) \subseteq F$ and $E(D_i, A) \cap F = \emptyset$;*
(ii) *if $d_F(v) \le f(v)$ for all $v \in V(G)$, then for any $v \in D$ there are both an even F-alternating trail and an odd F-alternating trail from the vertices of B_0 to v.*

Anstee [5] obtained a formula for the fractional f-factor number.

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Theorem 1.3 (Anstee [5]). *Let G be a graph and $f : V(G) \to Z^+$ be an integer-valued function on $V(G)$. Then*

$$def_f(G) = \max\left\{ f(T) - f(S) - \sum_{v\in T} d_{G-S}(v) \mid S, T \subseteq V(G), S \cap T = \emptyset \right\}.$$

Lemma 1.4. *Let H be a graph and $g : V(H) \to Z^+$ an integer-valued function. If H is a g-critical graph with at least three vertices, then H has a g-fractional factor.*

Proof. Let F be a g-optimal graph of H such that $d_F(v) \le g(v)$ for all $v \in V(G)$. Since H is g-critical, then $D = V(H)$ and $def(H) = 1$. Let $v \in V(H)$ such that $d_F(v) = g(v) - 1$. By Theorem 1.2(ii), there exists an odd F-alternating trail, say P, from v to v. Next we construct an indicator function h as follows:

$$h(e) = \begin{cases} 1/2 & e \in E(P), \\ 1 & e \in E(F) - E(P), \\ 0 & \text{otherwise}. \end{cases}$$

It is easy to check that h is a fractional f-indicator function of G. □

Now we present our main theorem of this work.

Theorem 1.5. *For any graph G, we have*

$$def(G) = \max_{S\subseteq V(G)} \left\{ f(T) - f(S) - \sum_{v\in T} d_{G-S}(v) \right\} + nc(G),$$

where $T = \{v \in V(G - S) \mid d_{G-S}(v) \le f(v)\}$.

Proof. Let F be an f-optimal subgraph such that $nc(F) = nc(G) = nc$ and $d_F(v) \le f(v)$ for all $v \in A$. Moreover, we may assume that $d_F(v) \le f(v)$ for all $v \in V(G)$, since if $v \in D_i$ and $d_F(v) = f(v) + 1$, then we can choose $e \in E(D_i) \cap E(F)$ incident with v such that $d_{F-e}(v) \le f(v)$ and $nc(F - e) = nc(F)$. Let $D_1, \ldots, D_{nc}$ be the F-near full components of $G[D]$. By Theorem 1.2(i), $E(D_i, B) \subseteq E(F)$ and $E(D_i, A) \cap E(F) = \emptyset$ for $i = 1, \ldots, nc$. Moreover, D_i has at least three vertices ($i = 1, \ldots, nc$) and $nc \le def(G)$.

Claim 1. $\max\{f(T) - f(S) - \sum_{v\in T} d_{G-S}(v)\} \le def(G) - nc$.

By Theorem 1.1, D_i is $f_B^{D_i}$-critical ($i = 1, \ldots, nc$). By Lemma 1.4, D_i has a fractional $f_B^{D_i}$-factor with indicator function $h_B^{D_i}$ ($i = 1, \ldots, nc$). Now we construct a function h on $E(G)$ as follows:

$$h(e) = \begin{cases} h_B^{D_i}(e) & e \in F_1 \cup \cdots \cup F_{nc}, \\ 1 & e \in E(F), \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$\sum_{e\in E(G)} h(e) = \frac{1}{2}\left(\sum_{v\in V(G)} f(v) - (def(G) - nc) \right).$$

By Theorem 1.3,

$$\begin{aligned} \mu_f(G) &= \frac{1}{2}\left(\sum_{v\in V(G)} f(v) - def_f(G) \right) \\ &= \frac{1}{2}\left(\sum_{v\in V(G)} f(v) - \left(\max\left\{ f(T) - f(S) - \sum_{v\in T} d_{G-S}(v) \right\} \right) \right) \\ &\ge \sum_{e\in E(G)} h(e) = \frac{1}{2}\left(\sum_{v\in V(G)} f(v) - (def(G) - nc) \right). \end{aligned}$$

Hence

$$\max\left\{ f(T) - f(S) - \sum_{v\in T} d_{G-S}(v) \right\} \le def(G) - nc.$$

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By Claim 1, it suffices to find a pair of disjoint subsets $S, T \subseteq V(G)$ such that $f(T) - f(S) - \sum_{v\in T} d_{G-S}(v) = def(G) - nc$. Let

$$B_1 = \{v \in B \mid d_F(v) < f(v)\}.$$

By Theorem 1.2, $E(B, B) \subseteq F$ and so $E(B_1, B_1) \subseteq F$. Moreover, $E(B_1, D) \subseteq F$. Otherwise, suppose $e \in E(B_1, D) - E(F)$; then $nc(F \cup e) > nc(F)$, a contradiction.

Let

$$B_2 = \{v \in V(G) \mid \exists \text{ an even } F\text{-alternating trail from a vertex of } B_1 \text{ to } v\},$$
$$A_1 = \{v \in V(G) \mid \exists \text{ an odd } F\text{-alternating trail from a vertex of } B_1 \text{ to } v\}.$$

Since the path of length zero can be seen as an even F-alternating one, we have $B_1 \subseteq B_2$.

Claim 2. $B_2 \cap D = \emptyset$.

Otherwise, suppose $v \in B_2 \cap D$ and there exists an even F-alternating trail, say P, from $u \in B_1$ to v. Then the component of $G[D]$ containing v is an F-full component. Otherwise, $F \bigtriangleup P$ is an optimal graph with $def_F(D_i) = 2$, a contradiction to Theorem 1.2. But then $nc(F \bigtriangleup P) > nc(F)$, a contradiction again. So Claim 2 is proved.

Claim 3. $A_1 \cap D = \emptyset$.

Otherwise, suppose $v \in A_1 \cap D$ and there exists an odd F-alternating trail, say P, from $u \in B_1$ to v. Similarly, the component of $G[D]$ containing v is an F-full component. By Theorem 1.1, $(F\triangle P)[V(D_i)]$ is an $f_B^{D_i}$-optimal graph and $d_{(F\triangle P)[V(D_i)]}(v) > f_B^{D_i}(v) > 0$. So there exists an edge $e \in (F\triangle P)[V(D_i)]$ incident with v. Then $nc((F\triangle P) - e) > nc(F)$, a contradiction.

By Claims 2 and 3, we see that $B_2 \subseteq B, A_1 \subseteq A$ and $E(B_2, (A-A_1)\cup B\cup D) \subseteq F$. Moreover, $E((B_2, (A-A_1)\cup B_2\cup D\cup C) \subseteq F)$.

Claim 4. *Every edge of F with one end in A_1 has the other end in B_2.*

Otherwise, let $e = uv \in F$, where $u \in A_1$ and $v \in (D \cup B) - B_2$. Since $u \in A_1$, there exists an odd F-alternating trail, say P, joining u to some vertex in B_1. If $e \notin E(P)$, then $P \cup e$ is an even F-alternating trail from a vertex of B_1 to v, a contradiction to $v \notin B_2$. So we consider $e \in P$. But then we have $v \in V(P)$ and $v \in B_2$, a contradiction again.

By Claims 2–4, we have $f(B_2) - f(A_1) - \sum_{v\in B_2} d_{G-A_1}(v) + nc = def(G)$. We complete the proof. □

From the proof of the above theorem, we can construct a function h with $h(e) \in \{0, \frac{1}{2}, 1\}$ only and thus obtain the following interesting consequence.

Corollary 1.6 ([6]). *For any graph G, let $f : V(G) \to Z^+$ be an integer-valued function. Then there exists a fractional indicator function h such that*

$$\sum_{e\in E(G)} h(e) = \mu_f(G),$$

where $h(e) \in \{0, \frac{1}{2}, 1\}$ for each edge $e \in E(G)$.

With Theorem 1.5, we are able to give an explicit formula for f-fractional numbers.

Corollary 1.7. *For any graph G, $\mu_f(G) = \frac{1}{2}(f(V(G)) - (def(G) - nc))$. In particular, $\mu_f(G) = \mu(G) + \frac{nc}{2}$.*

From the definitions, clearly $\mu_f(G) \geq \mu(G)$. But when does the equality hold? The next result gives a characterization for the family of graphs with $\mu_f(G) = \mu(G)$. Using F-alternating trails, the set $D(G)$ can be determined in polynomial time. Therefore, the graphs with the property $\mu_f(G) = \mu(G)$ can be identified efficiently.

Corollary 1.8. $\mu_f(G) = \mu(G)$ *if and only if* $D(G) = \emptyset$.

Proof. If $D(G) = \emptyset$, then $nc = 0$. By Corollary 1.7, the result follows. Conversely, let $\mu_f(G) = \mu(G)$; then $nc = 0$. Every component of $G[D]$ is F-full. Suppose $D \neq \emptyset$, and let $v \in D$. Note that there exists an alternating trail joining v to some vertex in B_1, a contradiction to the choice of F. □

Corollary 1.9. *If G is a bipartite graph, then $\mu_f(G) = \mu(G)$.*

Note that Corollaries 1.7–1.9 generalize the corresponding results given in [3] for fractional matchings.

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