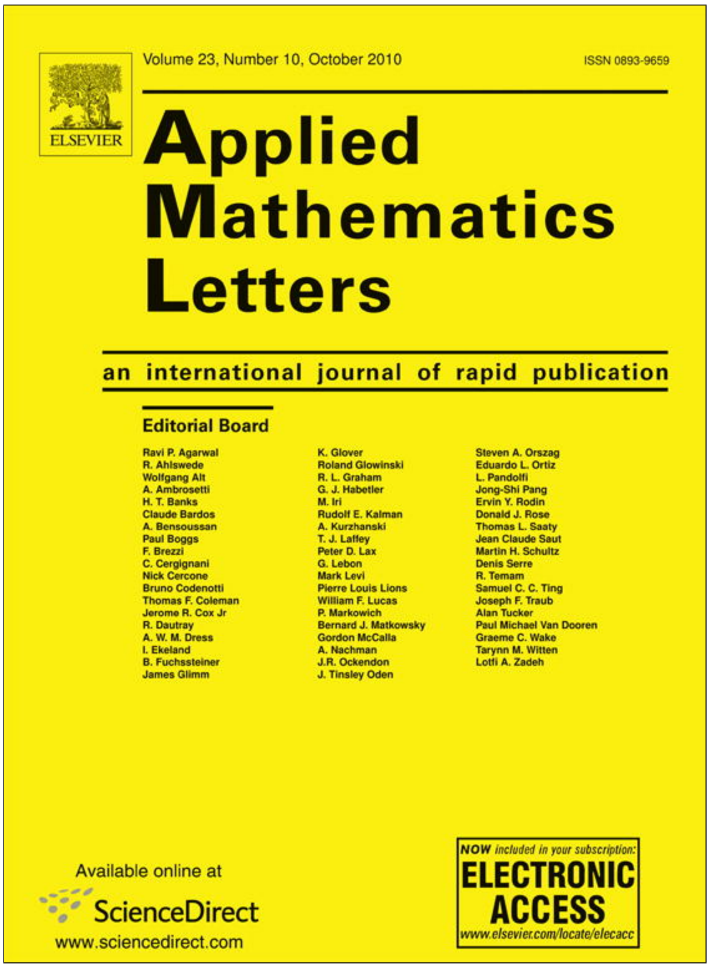
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A note on graph minors and strong products[☆]

Zefang Wu[a], Xu Yang[a,*], Qinglin Yu[b]

[a] Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin, China

[b] Department of Mathematics and Statistics, Thompson Rivers University, Kamloops, BC, Canada

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ABSTRACT

Let $G \boxtimes H$ and $G \Box H$ denote the strong and Cartesian products of graphs G and H, respectively. In this note, we investigate the graph minor in products of graphs. In particular, we show that, for any simple connected graph G, the graph $G \boxtimes K_2$ is a minor of the graph $G \Box Q_r$ by a construction method, where Q_r is an r-cube and $r = \chi(G)$. This generalizes an earlier result of Kotlov [2].

1. Introduction

Graphs considered in this note are finite, undirected, simple and connected. We use [1] for terminology and notation not defined here. The *strong product* $G_1 \boxtimes G_2$ of two graphs G_1 and G_2 has vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if (1) u_1 is adjacent to u_2 and $v_1 = v_2$ (we call it a *horizontal edge*); or (2) $u_1 = u_2$ and v_1 is adjacent to v_2 (we call it a *vertical edge*); or (3) u_1 is adjacent to u_2 and v_1 is adjacent to v_2 (referred to as a *type* (3) *edge*). For example, $K_2 \boxtimes K_2 = K_4$. The *Cartesian product* $G_1 \Box G_2$ of two graphs G_1 and G_2 is obtained from $G_1 \boxtimes G_2$ by deleting the 'type (3)' edges. For example, $K_2 \Box K_2 = C_4$. The well-known n-dimensional cube or *n-cube* Q_n can be viewed as the Cartesian product of n copies of $Q_1 = K_2$.

A graph H on vertex set $\{1, \ldots, n\}$ is a *minor* of a graph G, denoted by $H \preceq G$, if there are disjoint subsets $V_1, \ldots, V_n$ of $V(G)$ such that: (1) every V_i induces a connected subgraph of G; and (2) whenever ij is an edge in H, there is an edge between V_i and V_j in G.

Kotlov [2] initiated the study of the minor in products of graphs and proved the following result.

Theorem 1.1 (*Kotlov [2]*)**.** *For every bipartite graph G, the strong product $G \boxtimes K_2$ is a minor of $G \Box C_4$.*

Chandran and Sivadasan [3] studied clique minors in the Cartesian product of graphs. Later, Wood [4] and Chandran, Kostochka and Raju [5] continued the study of clique minors in a Cartesian product of graphs. In particular, Wood [4] showed that the lexicographic product $G \circ H$ is a minor of $G \Box H \Box H$ for every bipartite graph G and every connected graph H. In this note, we continue the study of the strong product minor in a Cartesian product started by Kotlov [2] and obtain several results in this direction.

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* Corresponding author. Tel.: +86 02223494039.
E-mail address: yangxu54@hotmail.com (X. Yang).

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2. Main results

Motivated by Theorem 1.1, we study minors in Cartesian products of graphs. The proof techniques are mainly constructive. As usual, χ denotes the chromatic number of G.

Theorem 2.1. *Let G be a connected graph with chromatic number χ. Then $G \boxtimes K_2 \preceq G\Box Q_\chi$.*

Denote the Hamming graph $K_{k_1}\Box K_{k_2}\Box\cdots\Box K_{k_d}$ with $k_1 = k_2 = \cdots = k_d = n$ by K_n^d. With a similar construction, we can obtain the following theorem.

Theorem 2.2. *Let G be a connected graph with chromatic number χ. Then $G \boxtimes K_n \preceq G\Box K_n^\chi$.*

Theorem 2.3. *Let G be a connected graph. Then $G \boxtimes K_2 \preceq G\Box K_a$, where a is an integer satisfying $\binom{a-1}{\lceil \frac{a}{2}\rceil} \geqslant \chi(G)$.*

Remark 1. If we choose a as small as possible (i.e., $a = \min\{m : \binom{m-1}{\lceil \frac{m}{2}\rceil} \geqslant \chi(G)\}$), the result is sharp when χ is small and G is sufficiently dense. For example, for any bipartite graph G which is sufficiently dense, $G \boxtimes K_2 \not\preceq G\Box K_3$ (see [2]). If $G = K_3$, then we have $K_3 \boxtimes K_2 \not\preceq K_3\Box K_3$, [1] but $K_3 \boxtimes K_2 \preceq K_3\Box K_4$.

What follows is an immediate corollary of the above.

Corollary 2.4. *For every 3-colorable graph G, the graph $G \boxtimes K_2$ is a minor of $G\Box K_4$.*

Hadwiger [6] linked the chromatic number of a graph G to the maximum size of its clique minor. He conjectured that every k-chromatic graph has a K_k-minor. This is one of the most intriguing conjectures in today's graph theory. The *Hadwiger number* $\eta(G)$ of a graph G is the maximum n such that K_n is a minor of G. A lot of research has been done on determining the Hadwiger number in special classes of graphs (see [3–5]).

Setting $G = K_\chi$ in Theorem 2.3, we readily obtain the following result on the Hadwiger number of a Hamming graph.

Corollary 2.5. *$\eta(K_\chi\Box K_a) \geqslant 2\chi$, if $\binom{a-1}{\lceil \frac{a}{2}\rceil} \geqslant \chi$.*

Remark 2. In [4], Wood proved that $\eta(K_n\Box K_m) \geqslant n\sqrt{\frac{m}{2}} - \mathcal{O}(n+\sqrt{m})$. It is not hard to verify that when $\chi \leqslant 35$,[2] Corollary 2.5 is an improvement of Wood's result.

3. Proofs of the main results

Before giving the proofs of main results, a few definitions and a lemma are required. They play important roles in the proofs of theorems. Let us call two partitions P, P' of the same set A *crossing* if every block of P intersects every block of P'. A partition containing k blocks is called a *k-partition*.

Lemma 3.1. *Let G, H be two graphs and $\chi = \chi(G)$. If there exist χ pairwise crossing n-partitions of $V(H)$ such that*

(P1) *every block of each partition induces a connected subgraph of $V(H)$,*
(P2) *every pair of blocks in a partition are adjacent (induce an edge with end-vertices in both blocks),*

then $G \boxtimes K_n$ is a minor of $G\Box H$.

Proof. Since G is χ-chromatic, there exists a χ-coloring c of $V(G)$ such that $c(v) = i$ when $v \in V(G)$ is colored i for all $1 \leqslant i \leqslant \chi$. Clearly, $\{v \in V(G) : c(v) = i\}$ induces an independent set in G for all $1 \leqslant i \leqslant \chi$. Suppose that $\{A_{i,1}, A_{i,2}, \ldots, A_{i,n}\}$, $1 \leqslant i \leqslant \chi$ are χ pairwise crossing n-partitions of $V(H)$ satisfying properties (P1) and (P2).

For each vertex $v \in V(G)$ and each $1 \leqslant j \leqslant n$, let

$$V_j(v) = \{(v, u) : u \in A_{c(v),j}\}.$$

Since for each i, $\bigcup_{j=1}^n A_{i,j} = V(H)$, then $\bigcup_{j=1}^n V_j(v)$ is an H-layer of $G\Box H$. And it is not difficult to show that the collection of sets $\{V_j(v) : 1 \leqslant j \leqslant n, v \in V(G)\}$ is a partition of $V(G\Box H)$. Now, we check that $G \boxtimes K_n \preceq G\Box H$ by definition.

For each $v \in V(G)$ and each $1 \leqslant j \leqslant n$, it follows from (P1) that $\{u : u \in A_{c(v),j}\}$ induces a connected subgraph in H, and hence $V_j(v)$ induces a connected subgraph in $G\Box H$ by the definition of the Cartesian product.

1 Suppose that $K_6 \preceq K_3\Box K_3$. Then $V(K_3\Box K_3)$ has branch sets $X_1, \ldots, X_6$, each of which is connected by at least one edge. If there exists X_i, say X_1, such that $|X_1| = 1$, then $\Delta(K_3\Box K_3) = 4$, contradicting the fact that X_1 is adjacent to X_i for all $2 \leqslant i \leqslant 6$. Thus, $|X_i| \geqslant 2$ and $\sum_{i=1}^{6} |X_i| \geqslant 12 > 9 = |V(K_3\Box K_3)|$, a contradiction.

2 If $a \leqslant 8$, then $\chi \leqslant 35$ and $2\chi \geqslant \chi\sqrt{\frac{m}{2}}$.

Consider an edge $e = (v_1, k)(v_2, l) \in E(G \boxtimes K_n)$. If e is of type (1) (a horizontal edge) or of type (3), then $v_1v_2 \in E(G)$. Clearly, $c(v_1) \neq c(v_2)$. Thus, $A_{c(v_1),k} \cap A_{c(v_2),l} \neq \emptyset$ by the assumption that these χ n-partitions are pairwise crossing. Assume $u_0 \in A_{c(v_1),k} \cap A_{c(v_2),l}$. Then $(v_1, u_0) \in V_k(v_1)$ is adjacent to $(v_2, u_0) \in V_l(v_2)$ in $G\Box H$. If e is of type (2) (a vertical edge), $v_1 = v_2$, $k \neq l$, it follows from (P2) that there is an edge $u_1u_2 \in V(H)$ where $u_1 \in A_{c(v_1),k}$ and $u_2 \in A_{c(v_2),l}$. So, $(v_1, u_1) \in V_k(v_1)$ is adjacent to $(v_2, u_2) \in V_l(v_2)$ in $G\Box H$. Hence, there is an edge connecting $V_k(v_1)$ and $V_l(v_2)$.

Therefore, there is a correspondence $G \boxtimes K_n \ni (v, j) \leftrightarrow V_j(v) \subseteq V(G\Box H)$. So, $G \boxtimes K_n$ is a minor of $G\Box H$. □

Now we are ready to prove the main theorems.

Proof of Theorem 2.1. By Lemma 3.1, we only need to find χ pairwise crossing bi-partitions of $V(Q_\chi)$ satisfying properties (P1) and (P2).

For $1 \leqslant i \leqslant \chi$, define

$$A_{i,0} = \{(j_1, j_2, \ldots, j_\chi) : j_i = 0 \text{ and } j_{i'} = 0 \text{ or } 1, i' \neq i\}$$

and

$$A_{i,1} = \{(j_1, j_2, \ldots, j_\chi) : j_i = 1 \text{ and } j_{i'} = 0 \text{ or } 1, i' \neq i\}.$$

Clearly, $\{A_{i,0}, A_{i,1}\}$, $1 \leqslant i \leqslant \chi$, are χ bi-partitions of $V(Q_\chi)$. Both $A_{i,0}$ and $A_{i,1}$ induce a graph isomorphic to $Q_{\chi-1}$. So, (P1) is true. Moreover, (P2) is obvious. Arbitrarily choose these two blocks from two different bi-partitions, say $A_{i_1,k}$ and $A_{i_2,l}$, where k, l are 0 or 1. Then

$$(0, \ldots, 0, i_1 = k, 0, \ldots, i_2 = l, 0, \ldots, 0) \in A_{i_1,k} \cap A_{i_2,l}.$$

Hence these χ partitions are pairwise crossing. This completes the proof. □

Proof of Theorem 2.2. Again by Lemma 3.1, we only need to find χ pairwise n-partitions of $V(K_n^\chi)$. Suppose $V(K_n) = \{1, 2, \ldots, n\}$. For every $1 \leqslant i \leqslant \chi$ and every $1 \leqslant j \leqslant n$, define

$$A_{i,j} = \{(l_1, \ldots, l_{i-1}, j, l_{i+1}, \ldots, l_n) : 1 \leqslant l_{i'} \leqslant n, i' \neq i\}.$$

It is easy to show that $\{A_{i,1}, A_{i,2}, \ldots, A_{i,n}\}$, $1 \leqslant i \leqslant \chi$, are χ pairwise crossing n-partitions of $V(K_n^\chi)$ satisfying properties (P1) and (P2). □

We proceed to the proof of Theorem 2.3.

Proof of Theorem 2.3. Without loss of generality, let $\{1, \ldots, a\}$ be the vertex set of K_a, where a is defined as in the assertion. Since $\binom{a-1}{\lceil \frac{a}{2} \rceil} \geqslant \chi$, we have at least $\chi = \chi(G)$ different bi-partitions $\{A_{1,1}, A_{1,2}\}, \ldots, \{A_{\chi,1}, A_{\chi,2}\}$ such that $1 \in A_{i,1}$ and $|A_{i,1}| = \lfloor \frac{a}{2} \rfloor$ for all i, $1 \leqslant i \leqslant a$.

Next, we prove that these χ bi-partitions are pairwise crossing and satisfy (P1) and (P2). Since K_a is a complete graph, it is obvious that (P1) and (P2) hold. Arbitrarily choose two blocks, say $A_{i,k}, A_{j,l}$, where k, l are 1 or 2, from different bi-partitions. We would like to show that $A_{i,k} \cap A_{j,l} \neq \emptyset$. Since $|A_{i,1} \cup A_{i,2}| = |A_{j,1} \cup A_{j,2}| = a$, $|A_{i,1}| = |A_{j,1}| = \lfloor \frac{a}{2} \rfloor$ and $1 \in A_{i,1} \cap A_{j,1}$, then $A_{i,1} \cap A_{j,1} \neq \emptyset$, so $A_{i,2} \cap A_{j,2} \neq \emptyset$ and $A_{i,1} \cap A_{j,2} \neq \emptyset$. Then we have $A_{i,k} \cap A_{j,l} \neq \emptyset$. This completes the proof. □

We include one more result of the same style as a conclusion of this note.

Proposition 3.2. *If G is a graph with chromatic number 4, then $G \boxtimes K_3 \preceq G\Box K_9$.*

Proof. By Lemma 3.1 again, it is sufficient to find four pairwise crossing tri-partitions of $V(K_9) = \{1, 2, \ldots, 9\}$ with properties (P1) and (P2). Since K_9 is a complete graph, (P1) and (P2) always hold for any tri-partition. On the other hand, we can always define a family $\{A_{i,j} : 1 \leqslant i \leqslant 4, 1 \leqslant j \leqslant 3\}$ of three-element subsets of $\{1, \ldots, 9\}$ such that

$$|A_{i,j} \cap A_{i',j'}| = \begin{cases} 1 & \text{if } i \neq i'; \\ 0 & \text{if } i = i', j \neq j', \end{cases}$$

as 3 is a prime (see, e.g., [5,7] for details). Then $\{A_{i,1}, A_{i,2}, A_{i,3}\}$, $1 \leqslant i \leqslant 4$, are four pairwise crossing partitions. □

Remark 3. In fact, Proposition 3.2 is better than the special case ($n = 3$) of Theorem 2.2. Moreover, when G is sufficiently dense, say $G = K_4$, we have $K_4 \boxtimes K_3 \preceq K_4\Box K_9$. It is a special case of $K_{p+1} \boxtimes K_p \preceq K_{p+1}\Box K_{p^2}$, where p is a prime. This result is widely known (see, e.g., [7]).

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