
Volume 23, Number 4, April 2010
ISSN 0893-9659
ELSEVIER
Applied Mathematics Letters
an international journal of rapid publication
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Applied Mathematics Letters 23 (2010) 385–389

Contents lists available at ScienceDirect

Applied Mathematics Letters

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Component factors with large components in graphs

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ARTICLE INFO

Article history:
Received 10 June 2009
Received in revised form 6 November 2009
Accepted 10 November 2009

Keywords:
Component factors
Isolated vertices

ABSTRACT

In this paper we obtain sufficient conditions using isolated vertices for component factors with each component of order at least three. In particular, we show that if a graph G satisfies $\text{iso}(G-S) \le |S|/2$ for all $S \subset V(G)$, then G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor, where $\text{iso}(G-S)$ denotes the number of isolated vertices in $G-S$.

1. Introduction

In this paper we consider component factors of graphs, which are defined as follows. For a set $\mathcal{S}$ of connected graphs, a spanning subgraph F of a graph G is called an $\mathcal{S}$-factor of G if every component of F is an element of $\mathcal{S}$. An $\mathcal{S}$-factor is also referred as a component factor. There have been many papers on component factors of graphs, but in most cases, $\mathcal{S}$ contains K_2 (i.e., a single edge), but it is relatively rare that $\mathcal{S}$ contains no small component. In addition, it is known that if $\mathcal{S}$ does not contain K_2, then in most cases finding a criterion for a graph to have an $\mathcal{S}$-factor is very difficult since finding a maximum $\mathcal{S}$-subgraph of a given graph is an *NP*-complete problem. In this paper we obtain several sufficient conditions in terms of the number of isolated vertices for a graph to have a component factor such that each component has order at least three.

We begin with some notation and definitions. We consider a finite simple graph G with vertex set $V(G)$ and edge set $E(G)$, which has neither loops nor multiple edges. We denote by $|G|$ the order of G. For a subset $S \subseteq V(G)$, $G-S$ denotes the subgraph of G induced by $V(G)-S$. For a vertex v of G, the degree of v and the neighborhood of v in G are denoted by $d_G(v)$ and $N_G(v)$, respectively. In particular, $d_G(v) = |N_G(v)|$. The minimum degree and the maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Denote by $\alpha(G)$ the independence number of G, which is the maximum cardinality among the independent sets of vertices of G. Let $\text{iso}(G)$ and $\text{Iso}(G)$ denote the number of isolated vertices and the set of isolated vertices of G, respectively. In particular, $\text{iso}(G) = |\text{Iso}(G)|$. For sets X and Y, $X \subset Y$ means that X is a proper subset of Y.

We denote the complete graph, the path and the cycle of order n by K_n, P_n and C_n, respectively. We denote the complete bipartite graph by $K_{n,m}$. A criterion for a graph to have a star-factor is given below.

Theorem 1 (*Amahashi and Kano [1]*). *A graph G has a star-factor, i.e., $\{K_{1,1}, \ldots, K_{1,n}\}$-factor, if and only if $\text{iso}(G-S) \le n|S|$ for all $S \subset V(G)$.*

A graph R is called *factor-critical* if for every vertex x of R, $R-x$ has a 1-factor (K_2-factor). A graph H is called a *sun* if $H = K_1$, $H = K_2$ or H is the corona of a factor-critical graph R with order at least three, i.e., H is obtained from R by adding a new vertex $w = w(v)$ together with a new edge vw for every vertex v of R (Fig. 1). A sun with order at least 6 is called a

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doi:10.1016/j.aml.2009.11.003

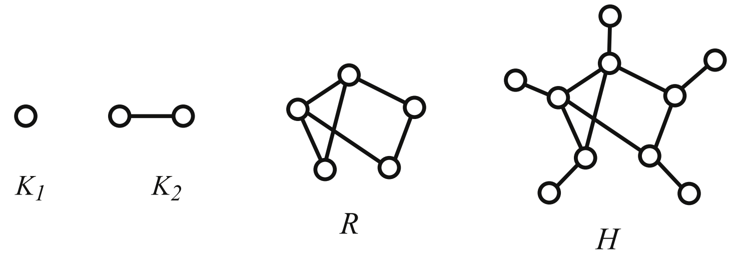

Fig. 1. A factor-critical graph R and the sun H obtained from R.

big sun. The number of sum components of G is denoted by $sun(G)$. The next theorem gives a criterion for a graph to have a path-factor each of whose components is of order at least three. Note that a shorter proof of the following theorem and a formula for a maximum $\{P_3, P_4, P_5\}$-subgraph of a graph was given in [2].

Theorem 2 (*Kaneko [3]*). *A graph G has a $\{P_3, P_4, P_5\}$-factor (i.e., $P_{\geq 3}$-factor) if and only if $sun(G - S) \leq 2|S|$ for all $S \subset V(G)$.*

In this paper we consider the following problem, and give partial answers to the problem.

Problem 1. Let G be a graph and λ be a positive rational number. If $\text{iso}(G - S) \leq \lambda|S|$ for all $\emptyset \neq S \subset V(G)$, what factor does G have?

2. Component factors with large components

In this section, we first prove the next theorem.

Theorem 3. *If a graph G satisfies*

$$\text{iso}(G - S) \leq \frac{2}{3}|S| \quad \textit{for all } S \subset V(G),$$

then G has a $\{P_3, P_4, P_5\}$-factor.

Proof. Suppose that G satisfies the condition but has no $\{P_3, P_4, P_5\}$-factor. By Theorem 2, there exists a subset $S \subset V(G)$ such that $sun(G - S) > 2|S|$. Assume that there exist a isolated vertices, b K_2's and c big sun components $H_1, H_2, \ldots, H_c$, where $|H_i| \geq 6$, in $G - S$. We choose one vertex from each K_2 component of $G - S$, and denote the set of such vertices by X. Then $|X| = b$. For each H_i, let R_i denote the factor-critical subgraph of H_i and let $Y_i = V(R_i)$. Then $\text{iso}(H_i - Y_i) = |Y_i| = |H_i|/2$. Let $Y = \cup_{i=1}^{r} Y_i$. So we have

$$\text{iso}(G - (S \cup X \cup Y)) = a + b + \sum_{i=1}^{c} \frac{|H_i|}{2}.$$

Moreover, it follows that

$$\begin{aligned} |S \cup X \cup Y| &< \frac{sun(G - S)}{2} + |X| + |Y| \quad (\text{from } sun(G - S) > 2|S|) \\ &= \frac{a + b + c}{2} + b + \sum_{i=1}^{c} \frac{|H_i|}{2} \\ &\leq \frac{3}{2}\left(a + b + \sum_{i=1}^{c} \frac{|H_i|}{2}\right) = \frac{3}{2}\text{iso}(G - (S \cup X \cup Y)). \end{aligned}$$

This contradicts the condition that $\text{iso}(G - S') \leq (2/3)|S'|$ for all $S' \subset V(G)$. ■

Let $m \geq 1$ be an integer Let $G = K_m + (2m + 1)K_2$, which is a graph obtained from K_m and $(2m + 1)K_2$ by joining every vertex of K_m to every vertex of $(2m + 1)K_2$. Then G has no $\{P_3, P_4, P_5\}$-factor. Let $T \subseteq V(G)$ be an independent set with $|T| \geq 2$. Then $T \subseteq V((2m + 1)K_2)$ and so $|N_G(T)| = |T| + m$. If $|T| \leq 2m$, then $i(G - N_G(T)) \leq 2|N_G(T)|/3$, otherwise $i(G - N_G(T)) = 2|N_G(T)|/3 + 1 = 2m + 1$. Since $\delta(G) \geq m + 1 \geq 2$, so $i(G - S) \leq 2|S|/3 + 1$ for all $S \subseteq V(G)$. Therefore the condition of Theorem 3 is sharp.

The next lemma is knows as Harlem Theorem, which is a generalization of Hall's Theorem.

Lemma 1. *Let G be a bipartite graph with bipartition (U, W), and $f : U \to \{1, 2, 3, \ldots\}$. If $|W| = \sum_{x \in U} f(x)$ and*

$$|N_G(S)| \geq \sum_{x \in S} f(x) \quad \textit{for all } \emptyset \neq S \subseteq U,$$

then G has a star-factor F such that each vertex u of U satisfies $d_F(u) = f(u)$, that is, every u is the center of a star $K_{1,f(u)}$ in F.

We next consider graphs satisfying $\mathrm{iso}(G-S) \le |S|/2$ for all $S \subset V(G)$.

Lemma 2. *If $|G| \le 6$ and $\mathrm{iso}(G-S) \le |S|/2$ for all $S \subset V(G)$, then G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.*

Proof. It is clear that if G satisfies the condition, then $\delta(G) \ge 2$ and $|G| \ge 3$. If $|G| = 3$, then G is connected and has a $K_{1,2}$-factor. If $|G| = 4$, then $\Delta(G) = 3$, which implies that G has a $K_{1,3}$-factor. Assume $|G| = 5$. If G has two non-adjacent vertices x and y, then $2 = |\{x, y\}| = \mathrm{iso}(G - (V(G) - \{x, y\})) \le |V(G) - \{x, y\}|/2 = 3/2$, a contradiction. Hence G is a complete graph K_5, and so it has a K_5-factor. Now we consider the case of $|G| = 6$. By Theorem 2, G has a $\{P_3, P_4, P_5\}$-factor, say F. Then F must be a P_3-factor, which is a $K_{1,2}$-factor. Therefore the lemma holds. ■

Theorem 4. *If a graph G satisfies*

$$\mathrm{iso}(G-S) \le \frac{|S|}{2} \quad \text{for all } S \subseteq V(G),$$

then G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

Proof. It is clear that $|G| \ge 3$ and $\delta(G) \ge 2$. Use induction on the lexicographic order of $(|G|, |E(G)|)$. So we assume that the theorem holds for a graph H with either $|H| < |G|$ or $|H| = |G|$ and $|E(H)| < |E(G)|$. Moreover, we may assume that G is connected and $|G| \ge 7$ by Lemma 2. Let

$$\beta = \min\left\{ \frac{|S|}{2} - \mathrm{iso}(G-S) \mid S \subset V(G) \text{ and } \mathrm{iso}(G-S) \ge 1 \right\}.$$

Then $\beta \ge 0$ as $\mathrm{iso}(G-S) \le |S|/2$. For a vertex x with $d_G(x) = \delta(G)$, we have $\beta \le |N_G(x)|/2 - \mathrm{iso}(G - N_G(x))$ and so

$$\delta(G) = d_G(x) = |N_G(x)| \ge 2(\beta + \mathrm{iso}(G - N_G(x))) \ge 2(\beta + 1). \tag{1}$$

Take a maximal vertex subset S such that $|S|/2 - \mathrm{iso}(G-S) = \beta$. Then

$$\frac{|S'|}{2} - \mathrm{iso}(G-S') > \beta \quad \text{for all } S \subset S' \subset V(G). \tag{2}$$

Claim 1. *$G-S$ has no component of order two or three.*

Assume that $G-S$ has a component D isomorphic to K_2. Let $V(D) = \{x, y\}$. Then

$$\frac{|S \cup \{x\}|}{2} - \mathrm{iso}(G - (S \cup \{x\})) = \frac{|S|+1}{2} - (\mathrm{iso}(G-S) + 1) < \beta,$$

a contradiction.

Assume that $G-S$ has a component D of order three. Let $V(D) = \{x, y, z\}$. Then

$$\frac{|S \cup \{x, y\}|}{2} - \mathrm{iso}(G - (S \cup \{x, y\})) = \frac{|S|+2}{2} - (\mathrm{iso}(G-S) + 1) = \beta,$$

a contradiction to the maximality of S.

Claim 2. *Every component D of $G-S$ with $|D| \ge 4$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.*

Let X be a non-empty subset of $V(D)$. Then by (2), we have

$$\frac{|S \cup X|}{2} - \mathrm{iso}(G - (S \cup X)) > \beta = \frac{|S|}{2} - \mathrm{iso}(G-S).$$

Thus $|X|/2 > \mathrm{iso}(D - X)$, which implies that D has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor by the induction hypothesis.

By Claim 1, let $G - S = aK_1 \cup \big(D_1 \cup \cdots \cup D_c\big)$, where $V(aK_1) = \mathrm{Iso}(G-S) = \{u_1, \ldots, u_a\}$ and each D_i is a component of $G-S$ with $|D_i| \ge 4$. It is immediate that

$$a = \mathrm{iso}(G-S) = |S|/2 - \beta \ge 1. \tag{3}$$

We construct a bipartite graph B with vertex set $V(B) = S \cup U$, where $U = \{u_1, u_2, \ldots, u_a\}$, such that two vertices $u_i \in U$ and $x \in S$ are adjacent in B if and only if u_i and x are joined by an edge of G.

Claim 3. *For every $\emptyset \ne Y \subseteq U$, we have $|N_B(Y)| \ge 2|Y| + 2\beta$, and $|N_B(U)| = 2|U| + 2\beta = |S|$.*

It follows from (3) and the choice of S that $|N_B(U)| = |S| = 2a + 2\beta = 2|U| + 2\beta$. Assume that there exists a subset $\emptyset \ne Y' \subset U$ such that $N_B(Y') < 2|Y'| + 2\beta$. Then, by the definition of β, $N_B(Y') = N_G(Y') \subset S$ satisfies

$$|Y'| \le \mathrm{iso}(G - N_G(Y')) \le \frac{|N_G(Y')|}{2} - \beta < |Y'|,$$

a contradiction. Hence the claim holds.

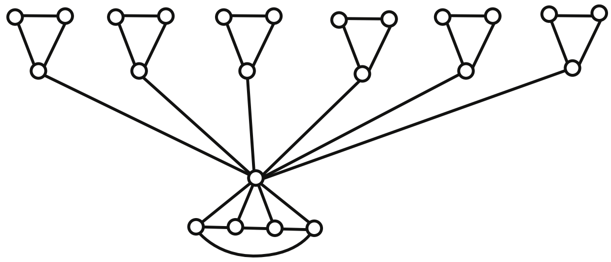

Fig. 2. A graph has no $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

Claim 4. *If $\beta \geq 2$, then the theorem holds.*

Assume $\beta \geq 2$. Then $\delta(G) \geq 6$ by (1). It is obvious that G has an edge e such that $G-e$ is connected. Let $X \subset V(G-e) = V(G)$. If $\mathrm{iso}(G - X) \geq 1$, then

$$\mathrm{iso}(G - e - X) \leq \mathrm{iso}(G - X) + 2 \leq \frac{|X|}{2} - \beta + 2 \leq \frac{|X|}{2}.$$

If $\mathrm{iso}(G - X) = 0$, then $\mathrm{iso}(G - e - X) \leq 2$. Further $\mathrm{iso}(G - e - X) \geq 1$ implies $|X| \geq 5$ as $\delta(G - e) \geq 5$. Hence if $\mathrm{iso}(G - X) = 0$, then $\mathrm{iso}(G - e - X) \leq 2 \leq |X|/2$. Therefore by the induction hypothesis, $G - e$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor, which is of course the desired factor of G.

From Claim 4 and the definition of β, it remains to consider the cases of $\beta \in \{0, 1/2, 1, 3/2\}$. Note that $|S| = 2|U| + 2\beta$.

Case 1. $\beta = 0$.

Define $f : U \to \{1, 2, 3 \ldots\}$ by $f(u) = 2$ for all $u \in U$. Then by Lemma 1 and Claim 3, B has a $K_{1,2}$-factor with centers in U. Hence by Claim 2, G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

Case 2. $\beta = 1/2$.

In this case, $|S| = 2|U| + 1$. Choose a vertex $u_1 \in U$ and define $f : U \to \{1, 2, 3, \ldots\}$ by $f(u_1) = 3$ and $f(u_i) = 2$ for all $u_i \in U - \{u_1\}$. Then $|N_B(Y)| \geq \sum_{x\in Y} f(x)$ for all $Y \subseteq U$ by Claim 3. Hence by Lemma 1, B has a $\{K_{1,2}, K_{1,3}\}$-factor. Therefore we can obtain a $\{K_{1,2}, K_{1,3}, K_5\}$-factor of G.

Case 3. $\beta = 1$.

Clearly, $\delta(G) \geq 4$ by (1). We consider two subcases.

Subcase 3.1. $|U| \geq 2$.

In this case, $|S| = 2|U| + 2$. Choose two vertex $u_1, u_2 \in U$ and define $f : U \to \{1, 2, 3, \ldots\}$ by $f(u_1) = f(u_2) = 3$ and $f(u_i) = 2$ for all $u_i \in U - \{u_1, u_2\}$. Then $|N_B(Y)| \geq \sum_{x\in Y} f(x)$ for all $Y \subseteq U$ by Claim 3. Hence, by Lemma 1, B has a $\{K_{1,2}, K_{1,3}\}$-factor and so G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.

Subcase 3.2. $|U| = 1$.

It is clear that $|S| = 2|U| + 2 = 4$ and $V(G) \neq S \cup U$ as $|G| \geq 7$. Let $H = G - (S \cup U)$, $U = \{u\}$ and $S = \{s_1, s_2, s_3, s_4\}$. Consider $G - \{s_1, u, s_2\}$. If $\mathrm{iso}(G - \{s_1, u, s_2\} - X) \leq |X|/2$ for all $X \subseteq V(G) - \{s_1, u, s_2\}$, then the theorem follows by the induction hypothesis. So we may assume there exists a subset $R \subseteq V(G) - \{s_1, u, s_2\}$ such that $\mathrm{iso}(G - \{s_1, s, s_2\} - R) > |R|/2$. However it follows that

$$\frac{3}{2} = \frac{|R| + 3}{2} - \frac{|R|}{2} < \frac{|R \cup \{s_1, u, s_2\}|}{2} - \mathrm{iso}(G - \{s_1, u, s_2\} - R) \leq \beta = 1,$$

a contradiction.

Case 4. $\beta = 3/2$.

By (1), we have $\delta(G) \geq 5$. Let $uv, vw \in E(G)$. Then for every $X \subseteq V(G) - \{u, v, w\}$ with $\mathrm{iso}(G - \{u, v, w\} - X) \geq 1$, it follows that

$$\mathrm{iso}(G - \{u, v, w\} - X) \leq \frac{|X \cup \{u, v, w\}|}{2} - \beta \leq \frac{|X|}{2}.$$

If $\mathrm{iso}(G - \{u, v, w\} - X) = 0$, then obviously $\mathrm{iso}(G - \{u, v, w\} - X) \leq |X|/2$. Hence by the induction hypothesis, $G - \{u, v, w\}$ has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor, which can be extended to a $\{K_{1,2}, K_{1,3}, K_5\}$-factor of G.

Consequently the theorem is proved. ■

We now show that the condition in Theorem 4 is sharp. Consider a graph G given in Fig. 2. Then G satisfies $\mathrm{iso}(G - S) \leq (|S| + 1)/2$ for all $S \subset V(G)$, but has no $\{K_{1,2}, K_{1,3}, K_5\}$-factor. Hence the condition of the theorem is sharp in this sense. The condition of Theorem 4 is sufficient but not necessary. For example, let $G = K_{1,3}$ (or C_{3m}, where $m \geq 2$). Then G contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor but dissatisfies the condition of Theorem 4.

Acknowledgements

The authors are grateful for the hospitality provided by the Center for Combinatorics at Nankai University, where this research was carried out. The first author is supported by Grant-in-Aid for Scientific Research of Japan and the third author is supported by Natural Sciences and Engineering Research Council of Canada.

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