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Toughness and $[a, b]$-factor with inclusion/exclusion properties*

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Abstract In this paper, we investigate the existence of $[a, b]$-factors with inclusion/exclusion properties under the toughness condition. We prove that if an incomplete graph G satisfies $t(G) \geqslant (a-1) + \frac{a}{b}$ and a, b are two integers with $b > a > 1$, then for any two given edges e_1 and e_2 there exist an $[a, b]$-factor including e_1, e_2; and an $[a, b]$-factor including e_1 and excluding e_2; as well as an $[a, b]$-factor excluding e_1, e_2 unless e_1 and e_2 have a common end in the case of $a = 2$. For complete graphs, we obtain a similar result.

Keywords $[a, b]$-factor, inclusion/exclusion property, toughness.

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1 Introduction

All graphs considered are simple and finite. We refer the reader to [1] for terminology and notation not defined here.

Let G be a graph. The degree of a vertex v in G is denoted by $\deg_G(v)$. For any disjoint subsets $X, Y \subseteq V(G)$, $E_G(X, Y)$ denotes the set of edges with one end in X and the other in Y and $e_G(X, Y) = |E_G(X, Y)|$. We use $E_G(X)$ to denote the set of edges with both ends in X.

For $X \subseteq V(G)$, the *neighbor set* of X in G, denoted by $N_G(X)$, is defined to be the set of all vertices adjacent to vertices in X. We use $G[X]$ to denote the subgraph induced by X.

For an integer-valued function f defined on a finite set X, we denote

$$f(X) = \sum_{x \in X} f(x), \quad f(\emptyset) = 0.$$

Given a function $f : V(G) \longrightarrow \mathbb{Z}^+$, we say that G has an *f-factor* if there exists a spanning subgraph F of G such that $\deg_F(v) = f(v)$ for every $v \in V(G)$. When $f(v) = k$ for all $v \in V(G)$, F is called a *k-factor*.

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Let g, f be integer-valued functions defined on $V(G)$. Then G has a *(g, f)-factor* if there exists a spanning subgraph F of G such that $g(v) \leqslant \deg_F(v) \leqslant f(v)$ for every vertex $v \in V(G)$. In particular, if $g(v) = a, f(v) = b$ for all $v \in V(G)$, F is called an *$[a, b]$-factor*.

If G is not complete, the *toughness* of G, $t(G)$, is defined by

$$t(G) = \min_S \left\{ \frac{|S|}{\omega(G - S)} \right\},$$

where the minimum is taken over all vertex cuts S of G, and $\omega(G)$ denotes the number of components in G. For complete graph K_n, we define $t(K_n) = n - 1$. A graph G is *k-tough* if $t(G) \geqslant k$.

Chvátal introduced the concept of toughness in [4], and mainly studied the relationship between toughness and the existence of Hamilton cycles and k-factors. He conjectured that every k-tough graph G ($k \in \mathbb{Z}^+$) has a k-factor if $k|V(G)|$ is even. Enomoto, Jackson, Katerinis and Saito [5] confirmed Chvátal's conjecture and showed that the result is sharp. Chen [2], Katerinis and Wang [7], Wang, Wu and Yu [11] studied the relationships between k-toughness of graphs and the existences of f-factors with various inclusion/exclusion properties.

As a generalization of Chvátal's conjecture, Katerinis [6] studied the relationship between toughness and the existence of f-factors, as well as $[a, b]$-factors. Katerinis proved the following theorem.

Theorem 1.1 (Katerinis [6])**.** *Let G be a graph and a, b be two positive integers with $b \geqslant a$. If $t(G) \geqslant (a-1) + \frac{a}{b}$ and $a|V(G)| \equiv 0 \pmod 2$ when $a = b$, then G has an $[a, b]$-factor.*

Later, Chen and Liu obtained a stronger result.

Theorem 1.2 (Chen and Liu [3])**.** *Let G be a graph and a, b be integers with $b \geqslant a \geqslant 2$. If $t(G) \geqslant a-1+\frac{a}{b}$ and $a|V(G)|$ is even when $a = b$, then for every edge e of G, there exists an $[a, b]$-factor containing e, and there exists another $[a, b]$-factor excluding e.*

In this paper, we consider the existence of $[a, b]$-factors with inclusion and/or exclusion of two edges in terms of toughness.

2 Preliminary Results

In order to prove the main theorems, we use the characterization of (g, f)-factors due to Lovász [9].

Theorem 2.1 ((g, f)-Factor Theorem)**.** *Let G be a graph and f, g be integer-valued functions defined on $V(G)$ such that $g(x) \leqslant f(x)$ for all $x \in V(G)$. Then G has a (g, f)-factor if and only if for all disjoint sets $S, T \subseteq V(G)$*

$$q_G(S, T) + \sum_{x \in T} \big(g(x) - \deg_{G-S}(x)\big) \leqslant f(S),$$

where $q_G(S, T)$ denotes the number of components C of $G-(S \cup T)$ such that $g(x) = f(x)$ for all $x \in V(C)$ and $e_G\big(T, V(C)\big) + \sum_{x \in V(C)} f(x) \equiv 1 \pmod 2$. (Hereafter, such a component C is called odd component.)

The lemma below can be deduced from Theorem 2.1.

Lemma 2.2 (Lam *et al.* [8])**.** *Let G be a graph, and g and f be two integer-valued functions defined on $V(G)$ such that $0 \leqslant g(x) < f(x) \leqslant \deg_G(x)$ for all $x \in V(G)$. Let E_1 and E_2 be two disjoint subsets of $E(G)$. Then G has a (g, f)-factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \emptyset$ if and only if for any disjoint subsets S and T of $V(G)$,*

$$\sum_{x \in T} \big(g(x) - \deg_{G-S}(x)\big) \leqslant f(S) - \alpha(S, T; E_1, E_2) - \beta(S, T; E_1, E_2),$$

where $U = V(G) - (S \cup T), \alpha(S,T;E_1,E_2) = 2|E_1 \cap E_G(S)| + |E_1 \cap E_G(S,U)|$ *and* $\beta(S,T;E_1,E_2) = 2|E_2 \cap E_G(T)| + |E_2 \cap E_G(T,U)|$.

In addition, we also need the following lemmas.

Lemma 2.3. *Let G be a graph and a, b be two positive integers with $b > a$. Suppose that there exists a pair of disjoint subsets S and T of $V(G)$ such that*

$$\sum_{x \in T} \big(a - \deg_{G-S}(x)\big) \geqslant b|S| - 3. \tag{1}$$

(a) Given S, if T is a minimal set with respect to (1), then $\deg_{G-S}(v) < a$ *for all* $v \in T$*;*

(b) given T, if S is a minimal set with respect to (1), then $\deg_T(v) > b$ *for all* $v \in S$*.*

Proof. As T is minimal with respect to (1), for any vertex $v \in T$,

$$\sum_{x \in T - v} \big(a - \deg_{G-S}(x)\big) < b|S| - 3. \tag{2}$$

Combining (1) and (2), we have $a - \deg_{G-S}(v) > 0$, i.e., $\deg_{G-S}(v) < a$.

Similarly, as S is minimal with respect to (1), for any vertex $v \in S$,

$$\sum_{x \in T} \big(a - \deg_{G-(S-v)}(x)\big) < b|S - v| - 3. \tag{3}$$

Combining (1) with (3), we have

$$e_G(S,T) - e_G(S - v, T) > b.$$

Thus $\deg_T(v) = e_G(S,T) - e_G(S - v, T) > b$. □

A subset I of $V(G)$ is an *independent set* of G if no two vertices of I are adjacent in G and a subset C of $V(G)$ is a *covering set* if every edge of G has at least one end in C.

Lemma 2.4 (Katerinis [6])**.** *Let G be a graph and $T_1, \ldots, T_{a-1}$ (T_j allows to be empty) be a partition of $V(G)$ such that $\deg_G(x) \leqslant j$ if $x \in T_j$. Then there exist a covering set C and an independent set I of $V(G)$ such that*

$$\sum_{j=1}^{a-1} (a - j)c_j \leqslant \sum_{j=1}^{a-1} (a - 1)(a - j)i_j,$$

where $|C \cap T_j| = c_j$ and $|I \cap T_j| = i_j$ for every $1 \leqslant j \leqslant a - 1$.

By the definition of toughness, we can easily show the following result.

Lemma 2.5. *Let G be an incomplete graph with toughness $t(G) \geqslant (a-1) + \frac{a}{b}$, where a, b are two positive integers with $b \geqslant a \geqslant 2$. Then $\delta(G) > a$. Moreover, if $a > 2$, then $\delta(G) > a + 1$.*

Proof. Since G is not complete, then $\delta(G) \geqslant 2t(G) \geqslant 2a - 2 + \frac{2a}{b}$. The conclusion follows directly. □

3 Main Theorems

We consider the inclusion and/or exclusion properties for complete graphs and incomplete graphs, respectively. We start with the case that G is an incomplete graph.

Lemma 3.1. *Let G be a graph with toughness $t(G) \geqslant a - 1 + \frac{a}{b}$, where a, b are integers satisfying $b > a \geqslant 2$. Let S, T be a pair of disjoint subsets of $V(G)$. If $S \neq \emptyset$ and $T \neq \emptyset$, then*

$$\sum_{x \in T} \big(a - \deg_{G-S}(x)\big) \leqslant b|S| - 4.$$

Proof. Suppose, to the contrary, that there exists a pair of disjoint subsets of $V(G)$, S and T with $|S|>0, |T|>0$ satisfying:

$$\sum_{x\in T}\big(a-\deg_{G-S}(x)\big)>b|S|-4.$$

By integrality,

$$\sum_{x\in T}\big(a-\deg_{G-S}(x)\big)\geqslant b|S|-3. \tag{4}$$

Moreover, suppose that S,T is a pair of minimal sets with respect to (4). Then by Lemma 2.3, for any vertex $x\in T$, $\deg_{G-S}(x)<a$ and for any vertex $x\in S$, $|T|\geqslant \deg_T(x)>b$ and so $|T|\geqslant b+1$.

For all i, $0\leqslant i\leqslant a-1$, define

$$T_i=\{x\in T:\deg_{G-S}(x)=i\}.$$

Denote $|T_0|=t_0$ and $G_0=G[T-T_0]=G[T_1\cup\cdots\cup T_{a-1}]$. Clearly, $\deg_{G_0}(x)\leqslant i$ for every $x\in T_i$. So, by Lemma 2.5, there exist a covering set C and an independent set I of G_0 such that

$$\sum_{j=1}^{a-1}(a-1)(a-j)i_j\geqslant\sum_{j=1}^{a-1}(a-j)c_j, \tag{5}$$

where $i_j=|I\cap T_j|$ and $c_j=|C\cap T_j|$ for all j, $1\leqslant j\leqslant a-1$. Clearly, We may assume that I is maximal in G_0. Moreover, we could assume that $I\cap C=\emptyset$ and $I\cup C=V(G_0)$. Note that $I\cup C=V(G_0)$ is followed from maximality of I and definition of covering sets. If $I\cap C\neq\emptyset$, set $C_0=C-I$. Clearly, the new set C_0 is still a covering set and $I\cup C_0=V(G_0)$. Now $|I|=\sum_{j=1}^{a-1}i_j\geqslant 1$. According to (4),

$$at_0+\sum_{j=1}^{a-1}(a-j)i_j+\sum_{j=1}^{a-1}(a-j)c_j\geqslant b|S|-3. \tag{6}$$

Let $Y=S\cup C\cup N_{G-S-T}(I)$. Then

$$|Y|=|S\cup C\cup N_{G-S-T}(I)|\leqslant|S|+\sum_{j=1}^{a-1}j\cdot i_j+|C|-e_G(C,I)$$

and $\omega(G-Y)\geqslant\sum_{j=1}^{a-1}i_j+t_0$. From the maximality of I, it follows that $|C|\leqslant e_G(C,I)$ and if the equality in $|C|\leqslant e_G(C,I)$ holds, then $\deg_I(x)=1$ for all $x\in C$. We claim that

$$|Y|\geqslant t(G)\cdot\omega(G-Y).$$

If Y is a cut set, we are done. Otherwise, $1\leqslant|I\cup T_0|\leqslant\omega(G-Y)=1$ and so $|I|=1$. Therefore, any vertex in C has at most one neighbor not in Y, and hence for every vertex $x\in T$, $|Y|\geqslant|S|+\deg_{G-S}(x)\geqslant \deg_G(x)\geqslant\delta(G)\geqslant t(G)$.

Next, we show the following claim.

Claim. $C\neq\emptyset$.

If $C=\emptyset$, then $|T|=t_0+|I|$. Since $|S|\geqslant|Y|-\sum_{j=1}^{a-1}j\cdot i_j\geqslant t(G)\cdot(t_0+\sum_{j=1}^{a-1}i_j)-\sum_{j=1}^{a-1}j\cdot i_j$ and $t(G)\geqslant a-1+\frac{a}{b}$, it follows from (6) that

$$at_0+\sum_{j=1}^{a-1}(a-j)i_j\geqslant(ba-b+a)t_0+\sum_{j=1}^{a-1}(ba-b+a-bj)i_j-3.$$

Then by $b>a\geqslant 2$ we have $a-1\leqslant(a-1)(|T|-t_0)\leqslant\sum_{j=1}^{a-1}(ba-b-bj+j)i_j\leqslant 3-(ba-b)t_0\leqslant 3-3t_0$. On the other hand, as $|T|\geqslant b+1$ and $b>a\geqslant 2$, we have $(a-1)|T|\geqslant 4$ if $t_0=0$ and $(a-1)(|T|-t_0)\geqslant a-1\geqslant 1>3-3t_0$ if $t_0>1$, which is impossible in either case. The claim is proved.

As $t_0 \geqslant 0$ and $b > a \geqslant 2$, there are several cases to consider.

Case 1. $t_0 = 0, b = 3$.

Note that when $b = 3$ and $a = 2$, for every vertex $x \in C$, since $x \in T$, $\deg_{G-S}(x) < a = 2$, and hence $\deg_I(x) = 1$.

We claim that $G[C]$ is either a singleton or a complete subgraph. Suppose there are two distinct nonadjacent vertices x_0, y_0 in C. Let $Y' = Y - \{x_0, y_0\}$. Since $\deg_I(x_0) = \deg_I(y_0) = 1$, we have $\omega(G - Y') \geqslant \sum_{j=1}^{a-1} i_j$ and $|Y'| \leqslant |S| + \sum_{j=1}^{a-1} j \cdot i_j - 2$.

We show that $|Y'| \geqslant t(G) \cdot \sum_{j=1}^{a-1} i_j$. If $\sum_{j=1}^{a-1} i_j > 1$, as $\omega(G - Y') \geqslant \sum_{j=1}^{a-1} i_j > 1$, Y' is a vertex cut. Hence, $|Y'| \geqslant t(G) \cdot \omega(G - Y') \geqslant t(G) \cdot \sum_{j=1}^{a-1} i_j$. If $\sum_{j=1}^{a-1} i_j = 1$, let $I' = \{x_0, y_0\}$ and $C' = T - I'$. Clearly, I' is independent in G_0 and $|I'| > |I|$, contradicting with the maximality of I.

Thus, $|S| \geqslant |Y'| - \sum_{j=1}^{a-1} j \cdot i_j + 2 \geqslant \sum_{j=1}^{a-1} \big(t(G) - j\big) i_j + 2$. Using (5), (6) and $t(G) \geqslant a - 1 + \frac{a}{b}$, we obtain

$$\sum_{j=1}^{a-1} (a-1)(a-j) i_j \geqslant \sum_{j=1}^{a-1} (ba - b - bj + j) i_j + 2b - 3 > \sum_{j=1}^{a-1} (ba - b - bj + j) i_j,$$

which is impossible, because $(a-1)(a-j) \leqslant ba - b - bj + j$ and $i_j \geqslant 0$ for all j, $1 \leqslant j \leqslant a-1$.

Now $|C| \leqslant \deg_{G-S}(x) < a = 2$ for every vertex $x \in C$. By Claim, $C \neq \emptyset$. Then $|C| = 1$. Let $Y'' = Y - C$. Clearly, $|Y''| \leqslant |S| + \sum_{j=1}^{a-1} j \cdot i_j - 1$, $\omega(G - Y'') \geqslant \sum_{j=1}^{a-1} i_j$ and $|Y''| \geqslant t(G) \cdot \sum_{j=1}^{a-1} i_j$. It follows from (6) and $|S| \geqslant \sum_{j=1}^{a-1} \big(t(G) - j\big) i_j + 1$ that

$$\sum_{j=1}^{a-1} (a-j) i_j + (a-1) \geqslant \sum_{j=1}^{a-1} (ba - b + a - bj) i_j + b - 3.$$

Therefore, $\sum_{j=1}^{a-1} (ba - b - bj + j) i_j \leqslant a - 1 = 1$. That is, $|I| = i_1 \leqslant 1$ and thus $|T| = |I| + |C| = 2$, a contradiction.

Case 2. $t_0 = 0, b > 3$ or $t_0 = 1, b = 3$.

We may assume that for any vertex $x \in C$, $\deg_I(x) = 1$. If there exists a vertex in C with at least two neighbors in I, then $|Y| \leqslant |S| + \sum_{j=1}^{a-1} j \cdot i_j - 1$. Thus, $|S| \geqslant t(G) \cdot (t_0 + \sum_{j=1}^{a-1} i_j) - \sum_{j=1}^{a-1} j \cdot i_j + 1$. Using (5) and (6), we have

$$\sum_{j=1}^{a-1} (a-1)(a-j) i_j \geqslant (ba - b) t_0 + \sum_{j=1}^{a-1} (ba - b - bj + j) i_j + b - 3 > \sum_{j=1}^{a-1} \big(ba - b - bj + j\big) i_j,$$

a contradiction.

Now, let $y_0 \in C$ and $Y' = Y - \{y_0\}$. Clearly, $|Y'| \leqslant |S| + \sum_{j=1}^{a-1} j \cdot i_j - 1$ and $\omega(G - Y') \geqslant t_0 + \sum_{j=1}^{a-1} i_j$ as $\deg_I(y_0) = 1$. Similarly, it is not difficult to show that $|Y'| \geqslant t(G) \cdot (t_0 + \sum_{j=1}^{a-1} i_j)$. Thus, $|S| \geqslant t(G) \cdot (t_0 + \sum_{j=1}^{a-1} i_j) - \sum_{j=1}^{a-1} j \cdot i_j + 1$.

Using (5) and (6) again, we have $\sum_{j=1}^{a-1} (a-1)(a-j) i_j > \sum_{j=1}^{a-1} \big(ba - b - bj + j\big) i_j$, a contradiction.

Case 3. $t_0 = 1, b > 3$ or $t_0 \geqslant 2$.

Note that $|S| \geqslant |Y| - \sum_{j=1}^{a-1} j \cdot i_j \geqslant t(G) \cdot (t_0 + \sum_{j=1}^{a-1} i_j) - \sum_{j=1}^{a-1} j \cdot i_j$ and $t(G) \geqslant a - 1 + \frac{a}{b}$. Therefore, according to (5) and (6),

$$\sum_{j=1}^{a-1} (a-1)(a-j) i_j \geqslant (ba - b) t_0 + \sum_{j=1}^{a-1} (ba - b - bj + j) i_j - 3.$$

If $t_0 \geqslant 2$ (resp. $t_0 = 1, b > 3$), as $b > a \geqslant 2$, then $(ba - b) t_0 \geqslant 2(ba - b) \geqslant 2b \geqslant 6$ (resp. $(ba - b) t_0 = ba - b \geqslant b > 3$), and thus $\sum_{j=1}^{a-1} (a-1)(a-j) i_j > \sum_{j=1}^{a-1} (ba - b - bj + j) i_j$, a contradiction.

The proof is complete. □

Now we are ready to prove the main theorem.

Theorem 3.2. *Let a,b be two integers with $b>a>1$ and $e_1=u_1u_2, e_2=v_1v_2$ be two distinct edges of an incomplete graph G. If $t(G)\geqslant(a-1)+\frac{a}{b}$, then G contains an $[a,b]$-factor including e_1 and e_2; and an $[a,b]$-factor including e_1 and excluding e_2; as well as an $[a,b]$-factor excluding e_1 and e_2 unless e_1 and e_2 have a common end in the case of $a=2$.*

Proof. Let E_1,E_2 be two edge sets (one of E_1 and E_2 is allowed to be empty) with $E_1\cup E_2=\{e_1,e_2\}$. The theorem holds if and only if G contains an $[a,b]$-factor F such that $E_1\subseteq E(F)$ and $E_2\cap E(F)=\emptyset$. Suppose, to the contrary, that G does not contain such an $[a,b]$-factor F. Then, by Lemma 2.2, there exists a pair of disjoint subsets S,T of $V(G)$ such that

$$\sum_{x\in T}\big(a-\deg_{G-S}(x)\big)>b|S|-\alpha(S,T;E_1,E_2)-\beta(S,T;E_1,E_2), \tag{7}$$

where $U=V(G)-S-T, \alpha(S,T;E_1,E_2)=2|E_1\cap E_G(S)|+|E_1\cap E_G(S,U)|$ and $\beta(S,T;E_1,E_2)=2|E_2\cap E_G(T)|+|E_2\cap E_G(T,U)|$.

On the other hand, as $t(G)\geqslant(a-1)+\frac{a}{b}$, by Theorem 1.1, G contains an $[a,b]$-factor. It follows from Theorem 2.1 that

$$q_G(S,T)+\sum_{x\in T}\big(a-\deg_{G-S}(x)\big)\leqslant b|S|. \tag{8}$$

By assumption $b>a$, we have $q_G(S,T)=0$.

Claim. $S\neq\emptyset$ and $T\neq\emptyset$.

Clearly, $S\cup T\neq\emptyset$. Otherwise, $\alpha(S,T;E_1,E_2)=0, \beta(S,T;E_1,E_2)=0$, and then (7) yields $\sum_{x\in T}\big(a-\deg_{G-S}(x)\big)>b|S|$, a contradiction to (8).

Case 1. $S=\emptyset$ and $T\neq\emptyset$. Then $\alpha(S,T;E_1,E_2)=0$. It follows from (7) and (8) that $\beta(S,T;E_1,E_2)\neq 0$. Then $E_2\neq\emptyset$, and hence either $E_2=\{e_2\}$ or $E_2=\{e_1,e_2\}$.

If $E_2=\{e_2\}$, according to (7), there exist two subsets $S=\emptyset$, $T\neq\emptyset$ of $V(G)$ such that $\sum_{x\in T}\big(a-\deg_{G-S}(x)\big)>b|S|-\alpha(S,T;E_1',E_2')-\beta(S,T;E_1',E_2')$ for $E_1'=\emptyset$ and $E_2'=\{e_2\}$. From Lemma 2.2, G contains no $[a,b]$-factors excluding e_2, a contradiction to Theorem 1.2.

Next assume $E_2=\{e_1,e_2\}$, which is the case of excluding e_1 and e_2. Then (7) becomes

$$\sum_{x\in T}\big(a-\deg_G(x)\big)>-2|E_2\cap E_G(T)|-|E_2\cap E_G(U,T)|.$$

If $a=2$, then $\delta(G)>a$ by Lemma 2.5. So, for any vertex $x\in T$, $\deg_G(x)\geqslant a+1$. Hence $-|T|\geqslant\sum_{x\in T}\big(a-\deg_G(x)\big)>-2|E_2\cap E_G(T)|-|E_2\cap E_G(U,T)|$. According to the relationship between E_2 and $E_G(T)$ or between E_2 and $E_G(U,T)$, there are several cases to discuss. If $E_2\subseteq E_G(T)$, then $|T|\geqslant 4$ because e_1,e_2 share no common ends when $a=2$ by assumption. But $-|T|>-2|E_2\cap E_G(T)|-|E_2\cap E_G(U,T)|=-4$, a contradiction. For other cases, we can deduce contradictions in similar ways.

If $a\geqslant 3$, then $\delta(G)>a+1$ by Lemma 2.5. Thus $\deg_G(x)\geqslant a+2$ for all $x\in T$. Therefore, $-2|T|\geqslant\sum_{x\in T}\big(a-\deg_G(x)\big)>-2|E_2\cap E_G(T)|-|E_2\cap E_G(U,T)|$, which is impossible.

Case 2. $S\neq\emptyset$ and $T=\emptyset$. Then $\beta(S,T;E_1,E_2)=2|E_2\cap E_G(T)|+|E_2\cap E_G(T,U)|=0$. It follows from (7) and (8) again that $\alpha(S,T;E_1,E_2)\neq 0$. Thus, $|E_1|\geqslant 1$, and then $E_1=\{e_1\}$ or $E_1=\{e_1,e_2\}$.

If $E_1=\{e_1\}$, from (7), we have $b|S|<\alpha(S,T;E_1,E_2)=2|E_1\cap E_G(S)|+|E_1\cap E_G(S,U)|\leqslant 2$, which is impossible since $b>a>1$.

If $E_1=\{e_1,e_2\}$, then $b|S|-2|E_1\cap E_G(S)|-|E_1\cap E_G(S,U)|<\sum_{x\in T}\big(a-\deg_{G-S}(x)\big)=0$. As $b\geqslant 3$, it follows that $3|S|<2|E_1\cap E_G(S)|+|E_1\cap E_G(S,U)|$. Note that $E_1=\{e_1,e_2\}$. There are several cases to discuss based on the relationship between E_1 and $E_G(S)$ or between E_1 and $E_G(S,U)$. We only consider the case of $E_1\subseteq E_G(S)$, i.e., $e_1\in E_G(S)$ and $e_2\in E_G(S)$. For other cases, the proofs go along the same line. If $e_1 \in E_G(S)$ and $e_2 \in E_G(S)$, then $|S| \geqslant 3$ and $3|S| \geqslant 9 > 4 = 2|E_1 \cap E_G(S)| + |E_1 \cap E_G(S,U)|$, a contradiction.

This completes the proof of the Claim.

Now as $S \neq \emptyset$ and $T \neq \emptyset$, it follows from Lemma 3.1 that

$$\sum_{x \in T}(a - \deg_{G-S}(x)) \leqslant b|S| - 4.$$

But $\alpha(S,T;E_1,E_2)+\beta(S,T;E_1,E_2) \leqslant 4$ and (7) reads $\sum_{x\in T}(a-\deg_{G-S}(x)) > b|S|-4$, a contradiction. □

We call a graph G an $[a,b]$-*graph* if $a \leqslant \deg_G(v) \leqslant b$ for all $v \in V(G)$. For almost regular graphs, Thomassen [10] proved the existence of all "almost regular factors".

Lemma 3.3 (Thomassen [10])**.** *If G is an $[r, r+1]$-graph, then G has a $[k, k+1]$-factor for all k, $0 \leqslant k \leqslant r$.*

Applying the above result, we obtain an inclusion/exclusion theorem for complete graphs.

Theorem 3.4. *Let a, b be two integers with $b > a > 1$ and $e_1 = u_1v_1, e_2 = u_2v_2$ be two distinct edges of a complete graph G.*

(a) If $t(G) \geqslant (a-1) + \frac{a}{b}$, then G contains an $[a,b]$-factor including e_1 and e_2;

(b) if $t(G) \geqslant a + \frac{a}{b}$, it contains an $[a,b]$-factor including e_1 and excluding e_2; as well as an $[a,b]$-factor excluding e_1 and e_2 if $V(e_1) \cap V(e_2) = \emptyset$.

Proof. Clearly, there exists a Hamiltonian cycle C_G, in G, which includes e_1 and e_2. Moreover, $G_1 = G - E(C_G)$ is a $(\delta(G)-2)$-regular graph. If $t(G) \geqslant a-1+\frac{a}{b}$, then $\delta(G) = t(G) \geqslant a$. As $0 \leqslant a-2 \leqslant \delta(G)-2$, by Lemma 3.3, G_1 contains an $[a-2, a-1]$-factor F_1. Then $C_G \cup F_1$ is a desired $[a,b]$-factor of G, including e_1 and e_2.

Since G is complete, $G - e_2$ contains a Hamiltonian cycle C'_G including e_1 when $|V(G)| \geqslant 5$. Now $G_2 = G - e_2 - E(C'_G)$ is a $[\delta(G)-3, \delta(G)-2]$-graph. As $t(G) \geqslant a + \frac{a}{b}$, $0 \leqslant a-2 \leqslant \delta(G)-3$ and G_2 has an $[a-2, a-1]$-factor F_2 by Lemma 3.3. Therefore, $C'_G \cup F_2$ is an $[a, a+1]$-factor, also an $[a,b]$-factor of G including e_1 and excluding e_2. If $|V(G)| = 4$, it is not difficult to show that G contains a $[2,3]$-factor including one edge and excluding another.

It remains to show that G contains an $[a,b]$-factor excluding e_1 and e_2. Since $V(e_1) \cap V(e_2) = \emptyset$, $G_3 = G - \{e_1, e_2\}$ is a $[\delta(G)-1, \delta(G)]$-graph. As $\delta(G) \geqslant a+1$, G_3 has an $[a, a+1]$-factor F_3 by Lemma 3.3. Trivially, F_3 is the desired $[a,b]$-factor. □

4 Remarks

The conditions in Theorems 3.2 and 3.4 are necessary for the conclusions.

Remark 4.1. The condition $a > 1$ is necessary. If $a = 1$, there exist many graphs with toughness $t(G) \geqslant \frac{1}{b}$ but containing no $[1,b]$-factors including/excluding one edge of G. Such examples can be found in [3].

Remark 4.2. When discussing the exclusion of e_1 and e_2, the condition $V(e_1) \cap V(e_2) = \emptyset$ is necessary, when G is complete or $a = 2$. Let $G = K_{a+2}$ and e_1, e_2 are two edges with a common end. It is easy to see that G contains no $[a,b]$-factor excluding e_1, e_2. For the case of $a = 2$, see Figure 1.

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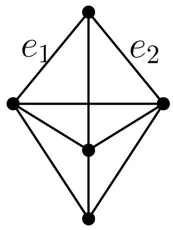

Figure 1: A graph with toughness $\frac{3}{2}$ contains no $[2, 4]$-factor excluding e_1, e_2

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